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THE MATHEMATICS TEACHER

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THE MATHEMATICS TEACHER

VOLUME XXII **FEBRUARY, 1929** NUMBER 2

GRAPHS AS A PART OF COOPERATIVE MATHEMATICS

BY FLETCHER DURELL

Belleplain, N. J.

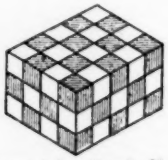
Symbols may be grouped in two main classes. Described in mathematical terms these classes or groups are (1) geometric, (2) algebraic.

Geometric symbols resemble the represented object as closely as possible. They depart from the visual or outward appearance reluctantly and as little as possible. This class of symbols or representations includes, for instance, photographs, pictures of all sorts, certain diagrams, and graphs. In a diagram like that given in the *Book of Knowledge* (Vol. 4, p. 1211) to show the four chambers of the human heart and the circulation of the blood, for the sake of making clear the essence of the matter, considerable changes have been made in the form and position of the parts represented, but as these changes are as slight as possible, the diagram is essentially geometric or pictorial.

Algebraic symbols, on the other hand, depart freely and aggressively from physical appearance. Their main purpose is not to represent the outside looks of an object or collection of objects, but their inner essence, and an important aim in devising these symbols is to make them of such a size and shape that they may be easily manipulated in accordance with some calculus, grammar, or system. In order to attain these ends likeness to visual or outward appearance is often entirely disregarded. Words (whether printed, written, or spoken), the number signs of arithmetic, the various symbols characteristic of algebra, chemical symbols, the signs used in short hand, and so on are examples.

The difference between these two principal kinds of symbolism

is illustrated in the two representations of the volume of a rectangular solid which follow:



Geometric Symbolism

$$V = lwh$$

or $V = 4 \times 3 \times 2$ cu. in.

Algebraic Symbolism

It is a familiar fact that each of the two kinds of symbols has its own special uses or valuable functions. Indeed, the characteristic values of each kind are so great and varied that it would require long and protracted study to form anything like an adequate conception of them.

Each of these two main kinds of symbolism also has defects or limitations peculiar to itself. Pictures and diagrams tend to limit us to outward appearance, often lack abstractness, and are difficult to treat in an extended, systematic way.

Correspondingly, algebraic symbols, especially when repeatedly and extensively used, are apt to become not only empty and meaningless, but even misleading, and their enlarged use a mere mechanical juggling. They may even be employed to deceive rather than to perform some useful function, such as in developing the subject matter represented and treated, and in aiding the reader to grasp this more fully and effectively.

Fortunately the two classes of symbols supplement each other and are thus fitted to cooperate. They aid each other both in positive and negative ways. They may be used together so as to multiply each other's values; also each is fitted to remedy the defects of the other. For instance, one of the most serious limitations of our number symbolism is that the digit signs in a number however large are all of the same approximate size and it is difficult to make ourselves constantly realize that, for example, the figure 6 in \$652.18 represents a value one hundred times as great as the 6 in \$6.45. But if we make a bar graph (or geometric symbolism) of the two sums mentioned, we are compelled to correct any erroneous conceptions which we have formed concerning the matter.

Hence, as a rule, the most interesting and also the most accurate and instructive books and articles are those which consist

both of verbal language and of pictures. The vast increase in the use of illustrations in connection with printed matter in recent times shows the growing appreciation of this fact.

The same holds true in other fields. Thus, the most interesting and in many ways a suggestive and instructive musical entertainment is that which combines both program (that is pictorial) numbers like Tschaikowsky's 1812, and also formal or classical elements, like some of Beethoven's compositions.

In mathematics when geometry and algebra (arithmetic being considered in this connection as mainly a branch of algebra) are correctly studied, they help in developing a grasp of the two fundamental kinds of symbolism. This is one of the principal values which may come from the study of mathematics. In this connection it is important to note that it is desirable to study cooperatively the two branches named, so that they shall not only supplement and reinforce each other, but also tend to produce a kind of underlying and general sense of symbolism.

The Graph

At this point one of the important sources of educational values in the graph becomes evident. In this special type (the graph), the two main kinds of symbolism are intertwined or interfused most cooperatively and fruitfully. Thus, in a graph, as a kind of geometric symbolism, we have areas and lines, as a kind of geometric symbolism, we have curves and other lines, Also on a graph, of symbols of the algebraic species, we use number signs, words, and often symbols like the dollar or percent signs.

In a graph each of the two principal kinds of symbols is essential to the other. The more efficient the one is, the more useful it makes the other.

Hence the right use of the graph constitutes an important instance of the cooperative use of geometry, algebra, and arithmetic, and all such cooperation, is a training in the most important and fundamental method of efficient living, and may be made a valuable result of the study of mathematics.

It is also worth while to note that this cooperative principle takes certain forms and certain special developments in connection with the graph besides the general one mentioned above.

In the following pages we shall make some study of these. In order to do this intelligently, we must first consider the direct practical values of the graph.

In this age where new arts and sciences are appearing, and when the world is gradually shaping itself into a new and more intimate international democracy, the tasks and opportunities of the educator for service grow correspondingly. There is an ever greater pressure for an intensification and an increase of efficiency in educational methods.

One of the most palpable and important of these new demands is for an education that shall be at the same time vocational and cultural. For many reasons, in recent years there has been a great increase in the number of forms of vocational training. Many attempts are being made to give this kind of education in such a way that it shall be not only directly practical but also broadly cultural. The topic of graphs is well fitted to illustrate the manner in which such a two-valued training may be achieved. For graphs not only have many direct practical uses, but also, if properly developed and taught, equally important disciplinary and cultural values.

Direct Practical Values

Taking up the vocational or directly practical values first, it is a familiar fact that long tables and elaborate arrays of numerical data, and much verbal discussion concerning them may often be condensed into a single graph or number-picture. By the use of this graph, the facts represented may be learned and grasped much more rapidly and in a more accurate and truthful form. Such reading also is often not only an economy to the mental powers of assimilation, but also a positive stimulus to the imagination and the reasoning powers.

Usually also a graph contains within itself certain laws and relations concerning the data represented, which are not evident at first glance. Hence such a number-picture frequently repays an intensive study. Thus in some ways a graph is found to be a kind of super-picture.

These direct practical values are obtained not merely in one's private life and work but, often also in one's social and political life, as when a person tries to get and keep an intelligent grasp

of certain social and governmental problems, the time and strength at one's disposal being limited.

As life becomes fuller and more complex the values and uses of such a highly efficient and approximately transcendent symbolism ever increase.

The extent of these useful functions becomes more striking and impressive when two or more lines occur on one chart, especially if these lines have causal or other important relations.

Other instances are diagrams on which three curves show the earnings, expenses and profits of a business, with perhaps another curve showing the amounts spent as dividends during a given period of time; a graphic chart showing the amounts of $1 at different rates of interest, both simple and compound; or one on which are curves of population, total wealth, and per capita wealth of a country; or a chart showing the exports and imports of a nation, with the difference between these represented by an area which may be negative for certain periods of time; and so on.

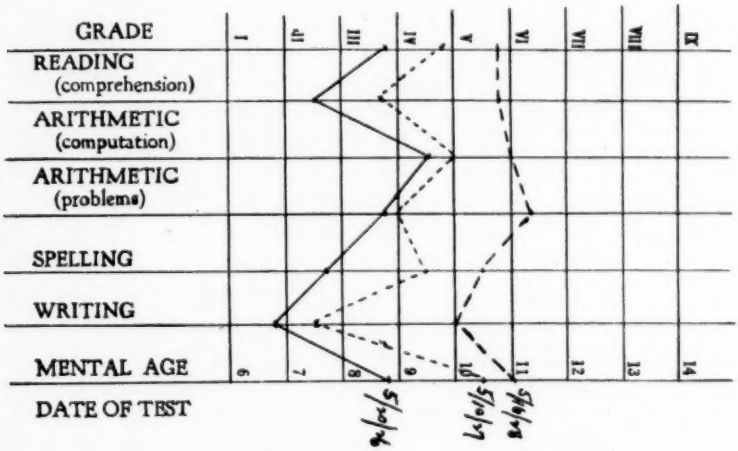
As corporations become larger and larger and the data to be mastered and carried in mind by their leading executives become increasingly numerous and complex, the values of the highly efficient form of symbolism we are discussing steadily increase, and the extensive use of graphs becomes in fact a necessity. A special part of their value is in aiding managers of large concerns to make quick and sound decisions, speed in such cases often being of vital importance. The ease with which facts are assimilated by the aid of graphic symbolism also leaves a manager free to use most of his mental energy in higher constructive thought as in devising new and better policies for the business under his charge.

Graphs are also very helpful in enabling a corporation to keep in close and sympathetic contact with its employees, stockholders, and the public, thus making the business a cooperative concern in a large way.

Hence it is not surprising to find graphic charts used more and more in the various reports of large corporations having a great number of stockholders like the American Telephone and Telegraph Company, and the United States Steel Corporation.

An individual pupil graph-record card has been devised on

which are recorded from year to year the results of the achievement and intelligence tests given near the end of each school year. One of these cards is here reproduced.



On this specimen card the record of a pupil is given only at the end of the third, fourth, and fifth grades. By inserting all of the other annual graph lines a superintendent will have a complete record of that pupil on the card for the entire period of his elementary training. Such cards showing the educational history of all the pupils in a county or other school district may be kept on file in a single cabinet. With practice the supervisor learns to read such cards almost at a glance and rapidly to decide the best course of treatment for exceptional cases. Other uses for such a file readily suggest themselves.

A business concern might file similar cards showing the records of its employees with reference to certain efficiency traits.

Some of the direct practical values of graphs are still broader in scope and character. Thus, lines, whether curved or straight, points, and areas are understood alike by all peoples whatever their nationality. Hence graphs form a kind of universal world language. Their use therefore tends to produce an international mind. The process is aided when the data graphed are also international or world-wide in scope. It is also aided by the fact that graphs show facts free from bias and prejudice and without distortion or unnecessary comment.

When educators study the underlying, permanent, and pivotal facts in each department of knowledge or life by graphic methods the different branches of education are fruitfully correlated. When graphs are used from the earliest stages of education through the highest, such use tends to unify the work done in the grades, the high school, and college, and thus to produce another form of educational cooperation.

Cultural Values

The above concludes our brief statement of the so-called vocational or directly practical values of the mastery of graphs. But, as has been indicated, the study of graphs also confers another set of benefits or skills more general, abstract, and underlying, often called cultural values. By careful attention to the matter, these formal and general values may be greatly increased. At the same time the vocational values will be enlarged and deepened, for it must always be remembered, that when rightly viewed and applied, cultural values are profoundly vocational and *vice versa*.

The first of these cultural values which we take up is a training in system and organization.

As we casually meet with graphs in our general reading, the different kinds seem almost endless in number and without any real systematic relation. Even a brief study shows that they are capable of an organized classification, based on the geometric concepts of the point, line (straight or curved), area, and volume, and the combinations of these. This matter will be treated more fully later.

For the present a more elementary classification will be sufficient. This covers the bar, circle, and line graphs, and the sub-species of each of these. Arranging them in tabular form we have the following:

I. *Bar Graphs.*

A. As to position of bars:

1. Horizontal. 2. Vertical.

B. As to number:

1. A single bar (representing some whole dissected into constituent parts).

2. Several bars.
   (1). Each plain (*i.e.* undissected).
   (2). Each dissected.
3. A set of multiple bars (as sales of two men) by months in two successive years.

II. *Circle Graphs.*
1. A single dissected circle.
2. Several circles.

III. *Line Graphs.*
1. A single line: (1) broken line, (2) curved line.
2. Several lines, broken or curved.

If a pupil works with a classification chart like this before him or in his mind, he is receiving an elementary, but valuable training in system and order.

If the pupil also works out and has before him in organized form a statement of the special advantages or useful peculiarities of each leading species of graphs, he will receive a still more advanced and valuable training from this study. Thus, for example, with horizontal bar graphs (as contrasted with vertical ones) it is usually easier to attach the verbal descriptive matter and it is more conveniently read. Vertical bars on the other hand are better adapted to represent data involving change or progress (*i.e.*, having an element of time) and also to illustrate the transition from a set of bar lines to a single curved or broken line. In like manner, a circle graph is especially adapted to show the relations of the component parts to a whole in a manifold way, as verbally, by figures, by percents, and by fractional parts. Similarly, for other special kinds of graphs.

Naturally there are convenient stages for gradually increasing the power of the pupil to use such classification and property charts. Thus, he may be asked to decide whether it is better to use a bar graph or a line graph for a given set of data; and later whether to use a bar, circle, or line graph; or to determine which species of bar graph is best in a given case.

As has been pointed out in another connection, the more extensive and complex the subject matter to be mastered by the pupil by our educational methods becomes, the greater is the need of handling it in some organized and systematic way and the greater will be the values which result. Similarly, as world life grows fuller and richer in complex details and relations, the

greater the importance of having powers to master and manage these in comprehensive organized ways. In spite of these facts, a regrettable process of desystematizing or even chaos seems to exist in many educational fields. Hence the importance of retaining the degree of system left in the processes of mathematical education and of adding to this if possible.

The study of graphs may also be made a means of developing the pupil's powers of initiative, invention, and resourcefulness, or what in general may be called his self-activity. The exercise of this faculty is called for in some degree in the choice of the special form of graph to be used in a given case as discussed above. The actual construction of individual graphs may be so taught as to develop the active, originating power in more vital and comprehensive ways.

For instance, in making line graphs, the most elementary stage of self-activity will be to give the pupil the axes with scales marked on them and also give him part of the graph line and ask him to complete that line. Progressively higher stages of initiative are required, for example, when only the axes with scales marked on them are given and the pupil is called upon to construct the entire graph line or lines; or when afterward the scales are given merely in sentence form; or when the pupil is asked to determine suitable scales for himself, first for small integral numbers and afterward for larger numbers which must be converted into round numbers.

The processes which have just been described are a means of developing a certain amount of original thought, power, or even reasoning in the pupil. The ability to do other kinds of reasoning can be inculcated by exercises in interpreting a graphic picture.

Thus, after a ready-made graph has been shown to the pupil, or he has made one to represent certain given facts, he may be asked to answer questions as to the information which may be obtained from the charts at hand.

Some of these questions may be simple, direct, and specific as when a pupil is asked to infer from the temperature curve for a given day the time at which the temperature was, say, 80°; or what the temperature was at 3:45 P.M.

A higher degree of reasoning power is called for and developed when the student is exercised in answering questions concerning

rates of change, maxima and minima, and causal relations between parts of a curve, or between two or more curves. A still higher stage of mental activity comes into play when a pupil is asked to deduce and state, say three important facts which may be learned from a given graph; or to state what he considers the most important fact which can be learned from a specified graphic chart; or to write a short essay on all the inferences which he can make from such a study.

It is plain that this progressive drill in extracting the utmost meaning from a graphic picture trains the youthful mind to reason, to infer, to deduce, and to imagine in vitally useful ways.

Again by practice in annexing the descriptive matter (or algebraic symbolism) to a graph a somewhat different kind of thought-power and a certain esthetic judgment are developed.

Thus, every graph should have a descriptive title and to make this title brief, adequate, and attractive often requires considerable study and discrimination. Sometimes it may be well to have two titles, a very brief and somewhat dramatic one above (perhaps in the form of a question to catch the eye) and a more complete one under the graph.

Sometimes it is well to insert on each bar of a bar diagram the number represented by this bar, while in other cases it is better to place these numbers by the side of the bars, to the right or left, with the numerical scale at the top of the chart.

On a circle graph, the possible descriptive items which may refer to each sector are the verbal description of the item, the number represented by the sector, and also the percent or fractional part of the whole. It requires taste and judgment to know which of these to use and which to omit in a given case.

On a line graph consisting of several lines or curves, the meaning of each line may be inserted next to the line on the diagram, or each line may have a number attached to it and an auxiliary table may show the meaning of these numbers.

Many other similar problems occur in special cases. Where verbal and numerical matter is to be attached to a graph the solution of these problems requires the exercise of thought, judgment, and good taste. In making a graphic chart it is important for the pupil to learn to see the chart as a whole, and to learn to adapt each part to this whole, so as to bring out the maximum of meaning in the most efficient way.

A few years ago a joint committee representing the leading engineering and scientific societies in the country and the Bureau of Standards at Washington published a report recommending certain fundamental standardized methods in making graphs for general use. This report included recommendations concerning matters such as ways of including the zero line on every graph; concerning the lines to be made heavy; the methods of inserting certain kinds of descriptive matter; the titles to be supplied; and so on. Books like Brinton's *Graphical Methods of Presenting Facts*[1] make other recommendations concerning the best methods of constructing graphs and charts. Brinton clarifies and illuminates his discussions by pointing out the violations of accepted principles and standards of graph construction in many specimens which have been published in otherwise excellent reports and articles.

Using the standard conventional methods already worked out as a basis and arriving at other like rules for himself, it is possible for the pupil to develop a kind of science or technique of graphing. This will include the choice, from kinds of paper ruled in various ways, of that particular variety most suitable in a given case; the use of different colors in drawing lines; and the making of some lines short-dotted or long-dotted as well as heavy or light.

Where on a given diagram, two curves are used along with a third resultant line derived from them, we may have a kind of addition or subtraction or even division of lines, the resultant in the last case expressing ratios or efficiencies between the date graphed by the other two curves. We thus approach a sort of calculus in the use of combinations of graph lines. In fact, may not the slide rule be regarded as a calculus in the use of a set of bar graph units?

All of these processes of forming and using a special technique of graphing, require and develop thought power of different kinds. Used in connection with the other cultural or reasoning processes outlined in the last few pages, their continued use should develop a special sense or faculty for making and using these number pictures, a kind of thought skill, as it were, in the use of symbolism.

[1] Published by the Engineering Magazine Company, New York City.

Other Values

Thus far we have discussed graphs as a useful or cooperative combination of the two main kinds of symbolism (which we have termed geometric and algebraic); the vocational values of good graphs; and lastly their formal or cultural values. The cooperative use of the two main kinds of sybolism in graphs, we have found to add to both their vocational and cultural values.

We next and finally consider certain other values of the graph in its relation to other subjects of study and educational processes.

Teachers of mathematics almost without exception have noted the strong interest and even joy aroused in pupils by the study of graphs. When these are studied in connection with the different branches of mathematics, this pleasure has important values, both direct and indirect. Thus, the study of arithmetic comprises the acquisition of many abstract and formal skills, and the process of acquiring these skills is often uninteresting and distasteful. The use of graphs judiciously sprinkled through the text relieves this difficulty.

Since algebra is more abstract than arithmetic and to a much larger extent consists of formal skills to be acquired, the judicious and systematic use of graphs in the study of algebra is even more valuable. For instance, it is an aid in helping the pupil to make the transition from arithmetic to algebra (perhaps the most difficult step in elementary, mathematical pedagogy); in giving a full grasp of the meaning and use of the formula, and in checking numerical results obtained from its use; in grasping the meaning of the function idea; in correcting the distortions necessarily involved in algebraic symbolism in particular cases and in general; and in maintaining and often increasing the interest of the pupil in his mathematical studies as a whole.

By this plan the study of the graph in connection with algebra need not be given in one or two separate chapters as is done in most texts, but the treatment of the topic could be broken up into certain progressive parts to be inserted at various places so as to arouse pervasive and cumulative interest. The joy and interest thus developed are of fundamental value in arousing the self-activity of the pupil and hence in creating a genuine class-room democracy.

The study of graphs in this cooperative way helps to give the pupil not only a general mathematical development but also a well-balanced and cultural one. Thus, some pupils are naturally picture-minded (or geometric), while others are word-minded (or algebraic). The right study of graphs is one way to correct this one-sidedness. The general cooperative study of algebra and geometry is a broader and more fundamental route to this result.

Recently much attention has been paid in schools to devising methods for the individual instruction of pupils and to organizing the subject-matter used in instruction, in ways adapted to three stages of mastery, or, as it is sometimes put, to three levels of ability. The desirability of this method of teaching pupils as a first step toward still more individual work is generally recognized, but difficulty has been experienced in organizing materials and methods of instruction in the way desired.

The topic of graphs lends itself admirably to this method of instruction and serves as a specimen or model to show how other topics may be treated. For the graph, because of its concreteness and sharply differentiated parts, is especially adapted to being presented in progressive stages. The more elementary of these have already been outlined on pp. 00–00.

It will be useful to indicate how the course in this topic may be enriched and developed for those pupils who are capable of going on to a thorough mastery of the subject and who desire to do so.

In the first place, there are many special kinds of graphs besides the three principal ones previously considered in this paper. Thus, we may have many kinds where points, areas (shaded or hatched in different ways), and even solids are involved, some of these tending to become special kinds of maps or three-dimensional models.

Among the special varieties are frequency graphs of various kinds, charts for objects moving in opposite directions, like those used in making out time-tables for railroad trains, distribution maps, pictograms, graphs composed of lines drawn on logarithmically-ruled paper and thus adapted to show rates or percents of increase or decrease. The last is of especial importance in this age when business and real progress are more and more

studied in terms of percents. Other forms will be found in treatises like those by Brinton or Haskell.[1]

All of these additional varieties of graphs form excellent subject-matter for advanced study by bright pupils. Such pupils may be asked to collect as many different species as possible and to group and classify them according to some comprehensive plan such as that suggested on p. 00; that is, by a method based on the point, line, area, and solid, and combinations of these.

The more capable pupils may also be asked to study and compare the relative merits of these additional species of graphs. A similar exercise would be to analyze and point out the defects of certain specimen graphs. Many of the popular graphs in current use violate one or more of the specifications for standardized practice agreed upon by the committee mentioned on p. 00. Thus, many of them violate the rule that a good line graph should always go to zero on its vertical scale. The more elaborate and complex graphs which should be the most useful are often the most defective and tend to create a prejudice against the whole graphic method.

The study of methods of filing graph cards (including the designing of cards adapted to filing), and the later use of such cards are also important. It is also possible to make the subject of graphic computation a part of the enriched course. Capable students may also be trained to design and redesign graphs until their results are artistic and attractive as well as expressive.

Thus, something like a graph sense may be developed which will not only be valuable in itself but will also be a part of the general symbolism instinct which is of a still higher order of value.

One day about twenty years ago Mr. Vail, the president of the American Telephone and Telegraph Company, sent to the department of accounts of that organization for a special report on a certain statistical matter. This report was prepared for him by a young man who had recently graduated from Harvard and who had come into the office as an ordinary clerk to learn the business from the ground up. The young man prepared the desired report largely in the form of graphic charts, a method somewhat new in business offices at that time. He also deduced

[1] "*How to Make and Use Graphic Charts,*" published by the Codex Book Company.

and annexed to the charts their meanings in the shape of percentages.

The report attracted the attention of Mr. Vail because of its simplicity and clarity as well as fulness of meaning. He sent for the young man and made him chief statistician of the company. A few years later when the United States entered the Great War, the young man was made the principal statistician of the American Government. Eight years later, at the age of 39, he became President of the American Telephone and Telegraph Company. In some ways this is the largest business organization in the world. It has more stockholders than any other company and has over 400,000 employees. It is worthy of note that today its annual reports to stockholders are made clear and simple by the abundant and efficient use of graphs.

The name of the man whose career has thus been briefly outlined is Walter Gifford. Of him a competent authority has said "no other man of his generation has risen so high in so short a time" in the business world.

In his appreciation of the young clerk's report, Mr. Vail not only showed that he was quick to recognize the business value of the new graphic instrument but he also perceived that in a man who could use these methods so well there must be unusual qualities such as enterprise, creative power, and organizing ability. The right study of graphs should help develop these powers to some degree in all pupils and thus make this topic an important element in cooperative mathematics. Such study will also serve as a specimen to show how a type of education may be developed which is simultaneously vocational and cultural. This is a higher kind of cooperative education.

The Ninth Educational Conference of the Ohio State University will be held on April 4, 5, and 6, 1928, at Columbus, Ohio. The mathematics section will have the following program:

"Beginning Plane Geometry with Similarity"—H. C. Christofferson, Miami University.

"Intuitive Geometry and Field Mathematics in the Junior High School"—Hale Pickett, Ohio University.

"The Activities of the Class Period in Mathematics"—Ernest R. Breslich, University of Chicago.

THE POINT SYSTEM OF TEACHING ALGEBRA

By BAILEY M. WADE

Principal, Junior High School, Union City, Tenn.

After several years of experimentation in teaching algebra to pupils of the ninth grade I have devised a plan which I shall refer to as the "point system." This plan has been highly satisfactory in the following ways:

1. It has been economical of the teacher's time.
2. It has served as a means of vitalizing the teaching of algebra to large classes of pupils.
3. It promotes independent work on the part of the pupils.
4. Each pupil at the end of the month can see how he stands in relation to the other members of his class, so that there is no question of getting an unfair mark.

The particular system referred to, briefly stated, is this: For the first ten minutes of each recitation period the pupils are given a number of problems which are intended to be a fair sampling of what the pupils should have learned from the day's assignment. The problems, taken from other texts, are arranged in order of difficulty. Enough problems are given so that the brightest pupil will barely finish in the time allotted. The tests are graduated so that the weakest pupil scarcely makes zero. Small yellow sheets are distributed as soon as the pupils enter the room. By using these sheets much waste of time is saved as well as confusion of tearing sheets of paper from tablets, and the like. Again this standard size is convenient in handling and scoring. At the end of the ten minutes pupils cease work, pass their papers in order to the end of the row where the class monitor collects them.

The problems are of the short answer type when possible so that the papers may be scored quickly. The pupil's score is the number of problems correct or perhaps two or five times the number correct since the instructor is the judge of the importance of each problem. It is possible to make five points each day. This makes the perfect score for the month one hundred points which may be used as a percentage grade. Naturally

on some days forty or more problems may be done in ten minutes in which case $\frac{1}{8}$ of a point would be the value of each. On another day not more than two problems might be solved in ten minutes; therefore each would carry a weight of $2\frac{1}{2}$ points.

After the papers are all in the instructor may well spend ten minutes in discussing common errors on the papers handed in the day previous. Usually it is better not to return the papers as the time will be prolonged in pointing out individual errors rather than group errors. After this the day's lesson is discussed. The instructor may do most of the work on the board assisted by suggestions from the class or pupils may solve the more difficult problems followed by explanation of the process. The socialized recitation is good for a change in which the instructor remains in the background while one of the better pupils demonstrates the principles of the day's lesson to the class.

The last ten minutes, at least, should be devoted to the assignment for the next day. This is more than an assignment time. It is a period of supervised study in which teacher and pupil approach the new task so that difficulties may be foreseen and errors prevented. The experienced teacher can do much in this short time to get the pupils ready for the next day's ten minute written lesson.

Some readers will think this method as time consuming when one hundred and fifty pupils are taught each day. Yet, it is almost incredible how short a time is consumed in scoring these little yellow sheets and in recording the scores in the class book. If pressed for time, the pupils may occasionally exchange papers and score the papers. Then the recording is all that has to be done except for an examination of a few papers for common errors to be discussed in class. The importance of correcting these errors before habits are formed cannot be over-emphasized. If the teacher's time is taken up with extra-curricular activities, there are always pupils who show exceptional interest and ability in algebra who would gladly score these papers. If a pupil is employed, it should be one who is not taking the course at this time.

At the end of the school month the number of points scored is computed by adding the total number of right answers for the twenty days. A distribution of the scores and the number

making each score is now made. The range may vary from 49 to 97 in a heterogeneous group. The range would not be very great where pupils are grouped homogeneously, as in the larger junior high schools of the country. In our own school this distribution is used.

A	B	C	D	F
10	20	40	20	10

The ten percent of the class making the highest scores are given A and those making the lowest scores F and so on. The following computations for a class of thirty pupils show how our marks are obtained:

No. of Pts.	97	96	94	91	90	88	86	85	84	81	79	74	70	68	67	65	60	58	55	49	Total
No. of Pupils	1	1	1	1	1	2	2	2	3	4	2	1	2	1	1	1	1	1	1	1	30
		A			B				C					D					F		

If the percent system were used, this particular case shows too large a number of failures unless the passing mark is very low.

The advantages of this point system of teaching algebra may be summed up as follows:

1. The scoring is strictly objective. The problems are either right or wrong. No credit is given for a good try.

2. The method is fair. No pupil will object to his mark as he can readily see that it is based upon the number of problems he solves. He can see how he stands in relation to other members of his class if the distribution is displayed each month.

3. There is no need of taking the roll at the beginning of the period as the yellow sheets check up on the presence or absence of the pupils.

4. Algebra teaching is vitalized. Pupils get pleasure in keeping their own scores and in watching their progress.

5. The short written lesson serves as a test for knowledge of the lesson, so that the remainder of the class may be given to teaching without recording grades for recitation work.

6. Pupils are brought to their tasks at once, thus eliminating the disorders which might occur at the beginning of a class period.

7. The method is economical of teacher time. Handing in of papers each day may be entirely eliminated so that there are no papers to be scored except the ten minute written quizzes.

TEACHING THE VERBAL PROBLEM IN INTERMEDIATE ALGEBRA

By BARNET RUDMAN

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None of the pitfalls of algebra is dreaded by the "average" pupil more than the verbal problem. In it he faces a veritable chaos—words, words, words—with no clue, no lead, no rule to show him the way out. And say what one may against the "rule" as a mathematical instrument one must credit it with giving the pupil a sense of security, a vehicle, as it were, to start him on his mental venture. It was no mere accident that the crossword puzzle, when in flower, had such a tremendous hold on the minds of both young and old. The originators of the puzzles were shrewd enough to supplement their problems with many helpful rules and numerous leads, thus achieving results that we, algebra teachers, had good reasons to envy. Equipped with a rule, even the inferior pupil will at least try to factor, or add, or multiply, or divide. He may misapply the rule and come to grief but he is spared a most depressing feeling, that of total helplessness. Yet with what confusion of mind, uncertainty, and lack of objective do pupils approach verbal problems of only medium difficulty! And with what disheartening results! Some get bewildered at the very "first reading" and quit. Others succumb to a positive intelligence gained in elementary algebra that they "can't do" verbal problems. And others still, willing and conscientious, labor hopelessly over a forlorn task striving vainly to bring order out of confusion with nothing, in the end, to remunerate their efforts save, perhaps, a new-born contempt for algebra, for the teacher and, in the case of the supersensitive pupil, for self. All this incompetence in a subject of major importance! For, if algebra can at all help in realizing the objective of teaching pupils to think clearly, the brunt of this objective must devolve upon the verbal problem.

It might seem that a subject so important yet so dreadful and trying to the pupil, would attract the best efforts of both teacher and textbook writer. Such, however, is not the case. The newer textbooks, to be sure, exhibit remarkable progress in the

choice of problem material, but little has been done in the direction of better methods of teaching the verbal problem. We guide our pupils cautiously through that part of algebra that is relatively "easy"—through the various types of factoring, the fundamental operations, the laws of exponents—but we plunge them head first into one set of verbal problems after another giving them nothing more in preparation for the ordeal than an illustrative problem or two. Now the illustrative problem as generally given in the classroom or textbook is of questionable value. At best it explains a solution discovered by the teacher but fails to teach pupils to make discoveries themselves. Consider, for example, the following problem and its solution from a reputable textbook:

> *To a mixture of 10 quarts which is 3/10 alcohol, I wish to add enough water so that the resulting mixture shall contain 20 per cent alcohol. How many quarts of water must be added?*

Hint: Let x = the number of quarts of water to be added.
$x + 10$ = the number of quarts in the mixture.
3 = the number of quarts of alcohol.

Therefore, $\frac{3}{x+10} = \frac{1}{5}$.

Granted that the above reasoning is thoroughly clear to the pupil, how is he to be guided by it in original work? If he forgets the steps of the solution, by what process of thinking may he recall them? The logic of the ready-made statement, $x + 10$ = the number of quarts in the mixture, may not be questioned, but what is likely to trouble the pupil is the origin of the statement, how did the teacher "happen to think of it"? The statement, $x + 7$ = the number of quarts of water in the mixture, is equally logical, nor is anything wrong with the logic of the allegation that $1/5x = 20$ percent of the water added. Why the author chose for his second step $x + 10$ = the number of quarts in the mixture is a genuine mystery to the average pupil. And in due time the whole matter assumes the appearance of a hit-or-miss game, with the teacher doing most of the hitting and the pupil most of the missing.

The numerous "word problems" that feature the newer textbooks, though answering a vital purpose, likewise fail to offer relief where relief is mostly needed—at the formulation of

a plan of solution. True, to be able to solve verbal problems a pupil must know how to answer questions of the type "if 7 pounds of cheese cost $1.89, what is the price of one pound? Of x pounds?" But he must also know when and how to ask such questions. It is indeed this asking of leading questions that provides the greatest opportunity for mental activity, and it is here that pupils mainly succeed or fail in their work on verbal problems. It follows that any method of teaching, not calculated to develop in the pupil this ability to ask himself the proper question at the proper time will fall short of its object.

Some teachers, to be sure, are aware of the confusion that verbal problems create in the minds of pupils and suggest this outline as a guide:

1. Read the whole problem carefully.
2. Represent the quantity to be found by x or some other letter.
3. If more than one quantity is to be found, represent the smallest quantity by x and the others in terms of x.
4. Translate the remaining statement of the problem into algebra in the form of an equation.
5. Solve the equation and check in the conditions of the problem.

This plan is very helpful in the simpler problems where statements are readily translatable into algebra, but, where the data is more or less involved requiring "interpretation" before it is made fit for an equation, the outline is inadequate. Problems from "life" in particular rarely yield to this mode of attack. How far, for instance, will a pupil who follows the above suggestions advance into this problem?

> *If 100 lb. of sea water contains $2\frac{1}{2}$ lb. of salt, how much water must be evaporated from 200 lb. of sea water in order that 12 lb. of the water remaining shall contain 1 lb. of salt?*

He will read the problem, take the second step—let x = the number of pounds of water to be evaporated, then what? There is water, there is salt, there is a mixture but where is the equation to be found? Again he is without guide and helpless.

What then is the remedy? The answer seems to point in the direction of the *Analytic Method* of solution. The analysis of a verbal problem like that of the geometry original should proceed backward from its objective. In geometry the pupil is taught to begin with what he wants to prove and by a succession of questions and answers to reach some known truth thus tracing a

proof which he can then present by the synthetic method. Will not the verbal problem yield to a similar attack? Let us return to the first problem quoted:

To find the value of x quarts of water to be added the pupil might proceed: "I could find a value for x, if I had an equation in x, since the equation is the tool I always use for the purpose. To write the equation I must discover two items that are equal to each other according to the conditions of the problem." Here he will pause to reread the problem in quest of a basis for his equation. He has made some advance and his interest, now awakened, is likely to carry him still further. An intelligent survey of the problem will reveal the only promising source of the desired equation in the statement" . . . so that the resulting mixture will contain 20 percent alcohol." The ratio 20 percent or 20/100 suggests the proportion:

$$\frac{?}{?} = \frac{20}{100}\text{; and, upon more thought,}$$

$$\frac{\text{quarts of alcohol}}{\text{quarts of mixture}} = \frac{20}{100} = \frac{1}{5}.$$

All that remains for the pupil to do is to replace "quarts of alcohol" and "quarts of mixture" by suitable arithmetic and algebraic symbols, and the equation is ready for solution. The original mixture contained $10 \times 3/10$ or 3 quarts of alcohol which amount remained unchanged. The entire mixture, however, has changed from 10 to $(10 + x)$ quarts. Hence if

x = the number of quarts of water to be added.
and 3 = the number of quarts of alcohol in the mixture.
then $10 + x$ = the number of quarts in the new mixture.

$$\therefore \quad \frac{3}{10 + x} = \frac{20}{100} = \frac{1}{5}.$$

Similarly in the second problem quoted above where it is required to find the value of x lb. of water to be evaporated from 200 lb. of sea water in order that 12 lb. of the water remaining shall contain 1 lb. of salt. (100 lb. of sea water contains $2\frac{1}{2}$ lb. of salt.) To find a value for x one needs an equation in x. What is the basis for such an equation? Apparently it is the statement ". . . in order that 12 lb. of water remaining shall contain 1 lb. of salt." Again the following proportion is suggested:

$$\frac{?}{?} = \frac{1}{12} \quad \text{or}$$

$$\frac{\text{lb. of salt}}{\text{lb. of solution}} = \frac{1}{12}$$

But $$\text{lb. of salt} = \frac{200}{100} \times 2\tfrac{1}{2} = 5$$

and $$\text{lb. of solution} = 200 - x$$

$\therefore$ $$\frac{5}{200 - x} = \frac{1}{12}.$$

The above scheme, of course, is not new. Teachers subconsciously discover solutions of problems by this same process of indirect analysis, but for some mysterious reason pupils are not generally made a party to the secret.

It must not be supposed that the writer parades this analytic method as a sort of a magic wand with which any pupil could touch any verbal problem and achieve a solution. It is his contention, however, that the analytic method, when properly taught, would make for more orderly thinking and cure some of the major ills surrounding the verbal problem. The formation of the equation will continue to trouble pupils. Only a fortunate few are directed to it by instinct but, where instinct is lacking, the extra sense can be developed, to some extent, by appropriate drill in analysis. Pupils should practice under supervision the reading of verbal problems and detecting parts or sentences that could serve as parts of foundations for equations. This calls for more time and a more systematic classification of problem material; for, the average pupil, to say nothing of the slower pupil, requires a good deal of drill on a single type. He may learn how to form the equation in a "mixture" problem but when next approaching a "digit" problem with its demand for a new line of reasoning, he may find himself unequal to the task. Worse still, he is likely to lose the path leading back to the "mixture" problem. Yet any "Second Course in Algebra" textbook devoting a whole exercise to the factoring of a trinomial of the type $x^2 + bx + c$ will exhibit ten or more different types in one exercise of 25 verbal problems. Is it any wonder that such exercises bewilder pupils?

It might be argued, and truthfully, that there are too many types of verbal problems to be studied as such separately. But suppose that five or six standard types were agreed upon by algebra teachers, and that they were taught thoroughly to all pupils as are the various cases of factoring, is it fair to assume, in the first place, that pupils would gain a higher degree of com-

petence in this vital department of our algebra course? Is it fair to assume, in the second place, that pupils would face new and miscellaneous problems with more self-reliance and less fear of failure? And is it fair to assume, in the third place, that fewer pupils would be stricken with the certainty that they "simply can't do verbal problems"? Leaving these as open questions for the reader to contemplate, the writer will now proceed on the dangerous mission of suggesting a tentative classification of types sufficiently troublesome to demand more systematic and thorough teaching. The analysis and discussion of typical problems will be attempted in connection with each class.

The Mixture Problem

Representatives of this family have already been considered elsewhere in the article. The equations, it was seen, are readily written in the form of proportions, which is true of most "alloy" and "mixture" problems. This is a valuable hint sparing pupils a good deal of uncertain and aimless effort. There are, however, problems in the class under consideration that do not yield equations in the proportion form. The following problem is typical:

> *If 100 lb. of water at 212° are mixed with 40 lb. at 32°, find the temperature of the mixture.*

Since there seems to be nothing in the problem to suggest a proportion nor even a basis for any other equation, the pupil might seek enlightenment in the physical aspects of the process. There he will discover that the amount of heat lost by the hot water equals the amount of heat gained by the cold water. To translate the above into algebra, the pupil will need a suitable unit of measure which, in the light of the present data, might best represent the amount of heat required to increase the temperature of 1 lb. of water by 1°. Hence, if x = temperature of the mixture, the hot water will lose $100\ (212 - x)$ units of heat while the cold water will gain $40(x - 32)$ units of heat, and $100(212 - x) = 40(x - 32)$.

The Distance Problem

This is a large class capable of many subdivisions. Prominent among them is the problem involving "handicap." For example,

A freight train started from a certain station and proceeded at the rate of 20 miles an hour. After $1\frac{1}{2}$ hours it was followed by a passenger train going 40 miles an hour. In how many hours will the second train overtake the first?

The analysis inquires into the possibilities for an equation. What in the problem may serve as the two equal items? Certainly not the rates of the two trains nor the lengths of time they spent on the trip. It must be the distances then, and the two trains indeed did cover the same distance. Hence, distance covered by freight train = distance covered by passenger train, or rate $\times$ time of freight train = rate $\times$ time of passenger train. If x = number of hours required by second train to overtake the first, then

$$\begin{aligned} &\text{Rate of freight train} &&= 20,\\ &\text{Time of freight train} &&= x + 1\tfrac{1}{2},\\ &\text{Rate of passenger train} &&= 40,\\ &\text{Time of passenger train} &&= x;\\ &\therefore \qquad 20(x + 1\tfrac{1}{2}) &&= 40x. \end{aligned}$$

The so-called "clock" problem falls easily into this group. For example,

When between 3 and 4 o'clock are the two hands of a clock together?

Here the minute hand labors under a handicap of 15 minutes. An analysis of the problem reveals the fact that the number of minutes covered by the minute hand = handicap + number of minutes covered by the hour hand. Letting x = no. of minutes after 3 o'clock when minute hand and hour hand are together,

$$\begin{aligned} x &= \text{no. of minutes covered by minute hand;}\\ 15 &= \text{handicap;}\\ \frac{x}{12} &= \text{no. of minutes covered by hour hand;}\\ \therefore\ x &= 15 + \frac{x}{12}. \end{aligned}$$

There is still another variety of distance problems of which the following may serve as an example.

A man can walk 4 miles an hour. How far may he walk into the country and ride back on a trolley car which runs 20 miles an hour, if he must be home in 3 hours?

Here one can find two possibilities for an equation:

(1) Distance into the country = Distance back home.
(2) Hours spent walking + hours spent riding on trolley = 3.

Using the latter equation and letting x = required distance,

$$\text{Hours spent walking} = \frac{x}{4},$$

$$\text{Hours spent riding} = \frac{x}{20};$$

$$\therefore \quad \frac{x}{4} + \frac{x}{20} = 3.$$

The Resultant-Rate Problem

Closely related to the distance problem is the one involving the rate of a body resulting from two forces acting upon it at the same time. For instance:

A steamer goes downstream 60 miles in 4 hours. The return trip takes two hours longer. What is the rate of the current and the rate of the boat in still water?

Analysis; x = rate of boat in still water.
y = rate of current.

Two equations in x and y are now needed. Apparently the rate of the boat downstream must serve as the source of one equation and that upstream as the source of the second equation. Or

(1) Rate of boat in still water + rate of current = rate of boat downstream.
(2) Rate of boat in still water − rate of current = rate of boat upstream.

But
rate of boat in still water $= x$,
rate of current $= y$,
rate of boat downstream $= 60/4 = 15$,
rate of boat upstream $= 60/6 = 10$;
$\therefore$ (1) $x + y = 15$,
(2) $x - y = 10$.

The Investment Problem

A man has $40,000 to invest and from it he wishes to obtain an annual income of $1800. If he invests part of his principal at 4 percent and the rest at 5 percent, how much must he invest at each of these rates?

In this problem the pupil should have no difficulty in recognizing the total investment as the foundation for the first equation and the total income as the foundation for the second equation. Thus if

x = investment at 4 percent,
y = investment at 5 percent;
then $.04x$ = annual income on first sum;
$.05y$ = annual income on second sum;
$\therefore$ (1) $x + y = 40{,}000$,
(2) $.04x + .05y = 1800$.

The Digit Problem

The formation of equations in this class is not difficult as a rule. It is in translating of statements into Algebra that trouble is mainly encountered. This weakness, however, is easily overcome by appropriate drill. Particular attention must be given to composition of numbers and expressing of numbers in terms of their digits both in direct and reverse order.

The Work Problem

Consider the following problem concerning work:

> *A and B together can complete a piece of work if A works 4 days and B 3 days or if A works 2 and B 6 days. How many days would it take each of them working alone to do the work?*

If x = the number of days for A to do the work alone, and y = the number of days for B to do the work alone, upon what ground could the two equations be written? Obviously A's and B's collective achievement, in both cases, representing the same unit of work must be broken up into their component parts. That is: (1) part of the task done by A in 4 days + part of the task done by B in 3 days = whole task = 1, and (2) part of the task done by A in 2 days + part of the task done by B in 6 days = whole task = 1. Now pupils can be led to see that if A does the work alone in x days, he will do in 1 day a fraction of the task amounting to $1/x$ and

in 4 days A will do a fraction amounting to $4/x$,
in 3 days B will do a fraction amounting to $3/y$,
in 2 days A will do a fraction amounting to $2/x$,
in 6 days B will do a fraction amounting to $6/y$;

$$\therefore \text{(1)}\ \frac{4}{x}+\frac{3}{y}=1,$$

$$\text{(2)}\ \frac{2}{x}+\frac{6}{y}=1.$$

This concludes what was intended as a very flexible and changeable classification of types of verbal problems upon which more emphasis and teaching effort might be concentrated.

Summary

In the foregoing article the writer sought to focus the attention of algebra teachers on the teaching of verbal problems and on

two reforms likely to prove beneficial. First, the analytic method of solution proposed as a helpful supplement to the synthetic method most generally used at the present time, and, second, a more systematized arrangement of problem material for purposes of increased thoroughness in the teaching of the subject. Should the article succeed in stimulating thought, not to say discussion, in this direction, its purpose would be amply served.

The mathematics section of the Texas State Teachers Association met at San Antonio in the Auditorium Annex of the First Baptist Church on Friday, November 30, at 9:00 A.M., with Mr. J. F. Howard presiding.

PROGRAM

Morning Session

1. Reading of minutes of last year's meeting.
2. Demonstration Lesson:
   MISS FRANCES DONECKER.
3. Use of Concept of Symmetry in Teaching of Geometry:
   DR. H. J. ETTLINGER.
4. Revision of Algebra Course of Study:
   MR. H. F. ALVES.
5. Review of Results of Inquiry on Requirements of Mathematics Teachers:
   DR. F. W. SPARKS.
6. Vitalizing Mathematics:
   DR. J. M. BLEDSOE.

Afternoon Session

1. Changes in Materials and Methods in Mathematics:
   DR. J. L. HENDERSON.
2. Curriculum Problems in Secondary Mathematics:
   DR. W. D. REEVE.
3. Teaching of Mathematics in Elementary Grades:
   GEORGE A. NEWTON.

GROUPING IN GEOMETRY CLASSES

By H. WEISSMAN

Franklin K. Lane High School, Brooklyn, N. Y.

The most important problem in our present system of mass education is that of adjusting instruction to suit the individual pupil. Our uniform "whole-class" method of instruction is designed for the "average pupil." The tendency to group together pupils of widely differing abilities brings about a situation where a lesson, which is too short for some, is too long for others. The more gifted pupil is deprived of the opportunity for his full development since he is not called upon to do his utmost, and therefore marks time. The less gifted pupil, on the other hand, being forced to compete with the brighter, becomes discouraged, develops the feeling of inferiority and habits of failure.

Individual instruction would be the ideal mode to meet this situation. In fact, it was the only prevalent mode up to the middle of the nineteenth century. But the advent of general education increased the number of pupils without a corresponding increase in the number of teachers, with the result that individual instruction became ineffective, since the teacher could not devote enough time to each pupil.

To remedy this situation methods of class instruction were evolved which proved far superior to the methods of individual instruction under the new conditions. However, the class method took no account of the individual differences of the pupils at all. Hence, different plans were evolved to get back to the individual and at the same time to accommodate the necessarily large numbers of pupils per teacher. Thus, there arose the Pueblo, the Batavia, and like plans. But these plans seriously lacked the social setting of the "whole-class" plan. The teacher became a task master while learning for the pupil became a process of doing one task after another, thus tending to defeat the cultural and social aims of instruction.

The plan outlined in this article offers a compromise between the purely individual plans and the whole-class procedure. It

is an outgrowth of a preliminary experiment conducted in the Franklin K. Lane High School in 1927. The results of this experiment led us to retain the plan in the beginners' classes in geometry in the Spring of 1928 and to continue it up to the present. The plan, with its mode of application and procedure, is presented in a rather detailed form, in the hope that other teachers may care to try it. The plan is based upon three main points.

1. The pupils are divided into three groups, *A*, *B*, and *C*, of different abilities, working in the same classroom under the same teacher.

2. Each group proceeds according to syllabi, differing both in quality and quantity of subject matter.

3. The grouping is not permanent for the whole term. The pupils may at certain definite periods be transferred from one group to another.

This method of grouping pupils was decided upon only after other means of dealing with individual needs were tried such as giving honor problems to bright pupils and supplementary assignments to the weaker pupils. The enthusiastic response of the pupils to this experiment suggested the idea of the classification into three groups. This was done after the mid-term examinations of the Fall term of 1927 upon the results of which the grouping was based.

The procedure in grouping adopted for the Spring term of 1928 which is now in use is as follows:

1. During the first five weeks, all the students are instructed according to the same syllabus as a single group. As soon as the classes are organized the pupils are informed in writing that they are later to be classified on the basis of their ability, originality, volunteer activity, and effort displayed. During this period the pupils are given a foundation in the subject. The teacher studies the individual pupils and tries to become as familiar as possible with the ability of each.

2. At the end of five weeks, the pupils are divided, on the basis of tests, in addition to the teacher's judgment, into the three groups. Candidates for a final average between 65 and 75 are put in the *C* group, those between 75 and 90 in the *B* group, and those between 90 and 100 in the *A* group. The pupils of the highest group were given their choice of group pro-

vided they felt that they did not care to do the extra work involved in belonging to the highest. Not one student, however, took advantage of this offer. On the contrary, the pupils asked for permission to join higher groups, promising to do the additional work even against the judgment of the teachers. This request was met by the offer of transfer to a higher classification in the near future if they qualified.

3. Each pupil receives a typewritten syllabus containing the work of all three groups. This enables those in the lower group to do the work of the next higher one having in mind the possibilities of transfer to the higher group.

Sample of Syllabus

Mathematics 3 Regular Syllabus—Second Quarter
The following exercises are required of the respective groups.
Based on Wells and Hart's *Modern Geometry*

C Group	*B Group*	*A Group*
96, 97, 101, (104), (106), 107, 109, 110, (112), 117, 119, (120), 121, 127 (129), (130), (144), (145), (146), (147), (149), (150), 151, 152, 153, (154), (157), 168, (169), (170). Page 274: 27, 28, 40, 41. Page 81: (221), (224).	96, 99, 101, (104), (106), 108, 109, 110, 112, 119, (120), 121, 127, (129), (130), (144), (145), (146), (147), (149), 150, 151, 152, 153, 154, 155, (157), 168, (169), (170). Page 81, (217), 220, (221), (224). Pages 272–275; 3, 7, 13, 19, 27, 28, 34, 38, 45, 46.	96, 99, 101, 103, (104), (106), 108, 110, (112), 119, 120, 121, 127, (129), (130), (144), (143), (146), (147), (149), (150), 151, 152, 153, 154, 155, 156, (157), 158, 168, (169), (170), 171. Page 81, (217), (218), 220, (221), (224). Pages 272–275, (3), 7, (8), 13, 14, 19, 27, 28, 32, (34), 35, (37), 38, 40, 41, 45, (47), (49).

The exercises enclosed in parentheses are to be written in notebooks after the pupils have recited on them in class. All exercises are to be written out on daily homework paper as indicated by the teacher or as indicated in the following schedule:

Lesson I. Study Par. 88–91, Prop. 13. Review Prop. 3, 12. A and B pupils write 272/3, 13.

Lesson II. Study Par. 93–97, Prop. 14, Review Par. 70, 71. Write Ex. 96, 97, 98, and Prop. 13. B pupils write 99, and 272/7. A pupils write same.

Lesson III. Study Prop. 15. Review Ex. 33, 36, Prop. 13. Write Ex. 101, 104, Par. 95, 97. A, B pupils write 273/8.

Lesson IV. Study Prop. 16, Par. 100–102. Review Prop. 15. Ex. 37. Write 106, 107, 108, Prop. 14. A and B write 273/14, 34.

Lesson V. Study Prop. 17, Par. 107–111. Review Prop. 5. Write 109, 110, 112, 117. Write Prop. 15. A pupils write 273/35.

Lesson VI. Study Prop. 18, 22. Review Prop. 8, Ex. 51. Write Prop. 177 and Ex. 119, 120, 127, 273/27. A and B write 273/37.

Lesson VII. Study Prop. 19. Review Ex. 62, 70, 76. Write Ex. 121, 130, 144, 145. Write Prop. 22 and Ex. 273/28. A and B write 81/217.

Lesson VIII. Study Prop. 20. Review Prop. 12. Write Ex. 146, 147, 149. Write Prop. 19. B write 81/221. A write 274/32.

Lesson IX. Study Prop. 21. Review Prop. 17. Write Ex. 150, 151, 152, Prop. 20. Write 275/41. B write 81/224. A write 81/224.

Lesson X. Study Prop. 23. Par. 126 and 130. Review Prop. 19. Write Ex. 153, 154, 168. B write 275/45. A write 275/47, 81/224.

Lesson XI. Study Prop. 24, 25. Review Prop. 13. Write Ex. 157, 169, 170, 81/224. B write 275/37. A write 81/218.

Lesson XII. Study Prop. 26, 27.

Put in notebooks, in addition to the exercises indicated in the table above, the following propositions: 13, 14, 15, 17, 19, 20, 21, 22, 23, 24, 25, 26.

The *C* pupils are responsible for the minimum syllabus consisting of the most important and necessary theorems and exercises, corresponding closely to the present Regents syllabus for New York State.

The *B* pupils are responsible for a number of more difficult theorems and exercises in addition to all of the *C* group work.

The *A* group in addition to most of the *B* and *C* work, is responsible for exercises of still greater difficulty.

The *A* and *B* groups study the same theorems added to those of the *C* group. These theorems are treated in the following manner:

They are assigned for home study to the *A* group first without any preliminary help or explanation on the part of the teacher and then presented by some of its members for the benefit of the rest of the class. The *B* group then studies them for the next recitation. The *C* group is responsible only for the fact knowledge of the special theorems and for their simple applications. Of course, the real divergence in the difficulty of theorems does not commence until the end of the first half of the term, the difference in syllabi being based mainly on the exercises. The differentiation in difficulty of the exercises, however, is apparent almost as soon as the grouping is accomplished.

Theorems not now included in the minimum Regents syllabus, such as the concurrency theorems, "The line joining the mid-points of two sides of a triangle," "The median of a trapezoid," etc., comprise the additional contents of the *A* and *B* syllabi. Among the exercises is: "The sum of the perpendiculars drawn from any point in the base of an isosceles triangle to the arms is equal to the altitude upon an arm."

To enable some pupils to proceed with the work ahead of the rest of the class, if they so choose, the work is arranged in daily lesson assignments and mimeographed for the benefit of both pupils and teachers. The teachers find these lesson assignments of great assistance. This is especially true of those who are teaching for the first time. The teachers, however, are not required to use them if they do not care to, but are responsible for the contents in the designated period of time. This syllabus brought the work up to the mid-term examination.

Sample of Syllabus after Mid-term Examination

Based on Wells and Hart's *Modern Geometry*

Mathematics 3 Regular Syllabus—Second Half

1. Study Par. 136–140, Ex. 177, 182, 199, 200. Write Ex. 178, 179, 181. B Ex. 226, A Ex. 228. Review Ex. 157.
2. Study Par. 141, Ex. 188, Par. 141. Write Ex. 180, 183, 184. B Ex. 201, A Ex. 193.
3. Study Par. 142, 143, 144, Ex. 189. Write Par. 141, Ex. 185, 192. B Ex. 200, A Ex. 159.
4. Study Par. 145, 170–172. Write Ex. 207, 208, 195, 197. B Par. 159, A Ex. 229. Review Par. 141.
5. Study Par. 151, 180–182. Review Ex. 204, 205, Par. 151. Write Ex. 196, 209, 210. B Ex. 237, Par. 159. A Ex. 232.
6. Study Par. 148–150, Par. 184, 185. Review Par. 151. Write Ex. (II) 11, 14. (I) 212, Par. 184, 185. B Ex. (I) 238. A Ex. Par. 161.
7. Study Par. 152, 153, 154. Review Par. 184, 185. Write (II) 17, 18, (I) 213, Par. 152, B Par. 161, A Ex. (I) 215.
8. Study Par. 189, 190. Review Par. 141. Write Ex. (II) 19, 21, Par. 189, B p. 273/15, A p. 273/8.
9. Study Par. 191. Review Par. 144. Write Ex. 214, 215 (II) 25, 29. (I) Par. 191. B Ex. (I) 280. A Par. 155.

10. Study Par. 192, review Par. 152. Write (II) Ex. 30, 31, 34, Par. 192. B Prop. 33, (I) 275. A p. 276/49.

11. Study Par. 194. Review Par. 192. Write Ex. (II) 36, 40, p. 282/5. B Ex. (I) 281, Par. 156, A Par. 156, Ex. (I) 276.

12. Study Par. 196–200. Review Par. 141. Write (II) Ex. 44, 48, 50. B Ex. (II) 164, A Ex. (I) 236.

13. Study Par. 195, 201, 202. Review Par. 151. Write Ex. (II) 51, 54, 55 and Par. 201. B Ex. 237/1, A Ex., Par. 163.

14. Study Par. 203. Review Par. 152. Write Ex. (II) 57, 109, (I) 225. B Par. 163. A Ex. (II) 52, p. 280/98.

15. Study 204–209. Review Par. 116. Write Par. 173, Ex. 64, 65, 111 (II). B Ex. (II) 53, A Ex. (II) 56, 58.

16. Study Par. 212. Review Par. 201. Write Ex. (II) 73, 74, 113, 115, Par. 212. B Ex. p. 280 (I), A Par. 164.

17. Study Par. 215, 213, 214. Review Par. 212. Write 75, 76, 79, 80, 118, Par. 215. B Ex. 77, Par. 164. A Ex. (I) 284.

18. Study Par. 216. Review Par. 215. Write Ex. 81, 82, 85, 86, 87, Par. 216. B Ex. (II) 89, A (I) 282.

19. Study Par. 217, 218, 219. Review Par. 216. Write 84, 90, 91, 92. B p. 285/51. A Par. 165.

20. Study Par. 220, 221. Review Par. 217, 218, 219. Write 93, 94, 96, 100 and Par. 221. B Par. 165, A Ex. (II) 101, 104.

21. Study Par. 222. Review p. 280/88. Write p. 285/49, 98, 103, 158 and Par. 222. B Ex. (II) 172, A Par. 166.

22. Study Par. 223, 224. Review p. 280/89–96. Write Ex. 210, 211, 106 and p. 283/16, B. Par. 166, A. Ex. (I) 285. A and B read Par. 2, 233, 235.

23. Study Par. 225–228, 237–239. Review Par. 189. Write Ex. 212 and Par. 240, Ex. p. 283/22, 26. B Ex. (II) 179, A Ex. (II) 166, 181.

24. Study Par. 191, 192. Write Ex. 213, 214, 191. B Ex. 164, 235, A Ex. 162.

25. Study Par. 201. Write Ex. 164, 192, 193, 213. B Ex. 195, A Ex. 173, 180. Honor Ex. 171.

General Reviews and Tests at Convenient Intervals

Sample of a Mid-term Examination

PLANE GEOMETRY-MATHEMATICS 3

Mid-Term Examination, April, 1928

Answer only those questions opposite which you find your group letter.

A	B	C	1	Prove that two lines are parallel if a transversal to these lines makes a pair of alternate interior angles equal.
	B	C	2	Prove that if two angles of a triangle are equal, the sides opposite these angles are also equal.
A			3	Prove that angles having their sides respectively perpendicular are either equal or supplementary.
		C	4	Prove that the altitude from the vertex of an isosceles triangle bisects the opposite side.
	B	C	5	If two lines bisect each other, prove that the lines joining their ends are parallel.
A	B		6	Prove that the perpendiculars drawn from two vertices of a triangle to the median of the side between these two vertices (or median prolonged) are equal.
A			7	Prove that the bisector of the vertex angle of an isosceles triangle bisects any line parallel to the base and included by the arms.
A	B	C	8i	Construct a triangle whose sides are 3, 4 and 4½ inches.
A	B	C	8ii	In this triangle, construct the median to the 4 inch side.
A	B	C	8iii	Construct the altitude to the 3 inch side.
		C	8iv	Bisect the angle opposite the 4½ inch side.
A	B		8v	Locate the point of intersection of the altitudes of the Δ.
		C	9i	Find the vertex angle of an isosceles triangle whose base angles are 50 degrees.
	B	C	9ii	AB is parallel to CD and XY meets AB and CD at E and F respectively. If angle CFY equals 50°, find angle FEB.
A	B	C	9iii	Two angles of a triangle are in the ratio 2 : 3, and the third angle equals 30°. Find the two angles.
A	B	C	9iv	Find the angle formed by the bisectors of a 50° angle and a 60° angle of a triangle.
A	B	C	9v	Find the base angle of an isosceles triangle if an altitude to an arm forms a 20° angle with the base.
A	B		9vi	In triangle ABC, AB equals AC and CD bisects angle C. If angle A equals 80° find angle BDC.
A			9vii	Median to AC in triangle ABC equals $\frac{1}{2}AC$. If angle C equals 40°, find angle A.
		C	10i	What is the construction line in proving that "The sum of the angles of a triangle is a straight angle"?
	B	C	10ii	What is the relation between the acute angles of a right triangle?
A	B	C	10iii	In proving "The opposite sides of a parallelogram are equal" why are the triangles congruent?
A	B	C	10iv	In proving "The diagonals of a parallelogram bisect each other" why are the triangles congruent?
A	B	C	10v	On straight line AB, points C and D are taken so that AC equals DB. Why is AD equal to CD?
A	B		10vi	In triangle ABC, angle B equals angle C, and BC is prolonged both ways forming exterior angles 1 and 2; why is angle 1 equal to angle 2?
A			10vii	What is the relation between the bisectors of the interior angles on the same side of a transversal to two parallels?

Check to see that you have answered questions as follows:

A pupils: 1, 3, 6, 7, 8i, ii, iii, v, 9iii, iv, v, vi, vii, 10iii, iv, v, vi, vii

B pupils: 1, 2, 5, 6, 8i, ii, iii, v, 9ii, iii, iv, v, vi, 10ii, iii, iv, v, vi

C pupils: 1, 2, 4, 5, 8i, ii, iii, iv, 9i, ii, iii, iv, v, 10i, ii, iii, iv, v

Sample of a Final Examination

FINAL EXAMINATION, JUNE, 1928

Mathematics 3 Regulars

C pupils answer questions 1, 2, 3, 8, 11, 12, 16, 17, 19.
B pupils answer questions 3, 4, 7, 9, 13, 14, 16, 18, 20.
A pupils answer questions 5, 6, 7, 10, 13, 15, 16, 18, 21.
Credits allowed 15, 15, 15, 10, 10, 10, 10, 10, 5.

Theorems

1. If the opposite sides of a quadrilateral are equal, the figure is a parallelogram.

2. An angle formed by two chords meeting at a point within a circle is measured by one half the sum of the arcs intercepted by the angle and its vertical angle.

3. Tangents drawn to a circle from the same point are equal.

4. In the same circle, equal chords are equally distant from the center.

5. If two angles of a triangle are unequal the side opposite the greater angle is the greater side.

6. If three or more parallels intercept equal segments on one transversal, they intercept equal segments on any transversal.

7. The bisectors of the angles of a triangle are concurrent at a point which is equidistant from the sides of the triangle.

Originals

8. If the altitudes to two sides of a triangle are equal, the triangle is isosceles.

9. If two chords drawn through any point on a radius of a circle make equal angles with the radius, the chords are equal.

10. If the diagonals of an inscribed quadrilateral are equal, the figure is an isosceles trapezoid.

Constructions

11. Construct a right triangle given the hypotenuse, and one acute angle.

12. Construct 2/3 of a given line.

13. Find all points at a given distance from a given point and equidistant from two given intersecting lines.

14. Construct triangle *ABC* given h_b t_b *B*.

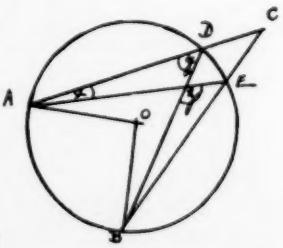
15. Construct triangle *ABC* given h_a t_a *b*.

Numerical

16. See accompanying figure.

Given: $\angle X = 20°$.
$\angle AOB = 120°$.

Find: Arcs *DE*, and *AB*.
Angles *C*, *Y* and *Z*.



17. Find each angle of a 10 sided equiangular polygon.

18. Given the sum of the interior angles of a polygon is 1080 deg. Find the number of sides.

Originals

19. If two equal chords are drawn from the ends of a diameter, on either side of the diameter, the chords are parallel.

20. If from any point in the circumference of a circle a chord and a tangent are drawn, the perpendiculars to them from the mid-point of the lesser arc are equal.

21. The circle drawn on one of the equal sides of an isosceles triangle as a diameter bisects the base.

It will be noticed that some of the questions in the mid-term examinations were to be answered by all the groups, the others only by the *A*, the *B* or the *C* group. In this manner, the *A* and *B* groups were checked for their knowledge of the *C* group work also.

Obviously, one criticism that might be leveled against the *A-B-C* mid-term examination is that it is unfair to give the same mark to a *C* pupil and an *A* pupil on examinations of unequal difficulty. This is a just criticism when we are comparing the achievement of a pupil with the achievements of all pupils in the same grade. This view also governs certification for graduation and college entrance. Some sort of a table, therefore, had to be designed which would convert more or less empirically, the marks of all groups into uniform standards of achievement. This we tried to do by means of the conversion table shown on following page.

The mark obtained on the examination is converted into a mark given to the pupil on his report card. The purpose of this conversion is to evaluate the work of each student:

1. As a member of his own group.
2. In terms of the maximum work of the *A* student.

It will be noticed for instance that a *C* pupil receiving a mark between 80 and 100 on the *C* examination, has his mark converted into 75–82 percent which puts him into the *B* classification. An *A* student, on the other hand receiving a mark of 60–55 percent on the *A* examination receives a converted mark of 80–75 percent which reduces him to the *B* group. A mark of 50 percent on the *A* examination puts him into the *C* group with a grade of 65 percent; a mark lower than 50 percent on his examination places him as a *C* (failing) pupil.

CONVERSION TABLE

If your Mid-term Mark is-	On the "A" Examination your		On the "B" Examination your		On the "C" Examination your	
	Report Card Mark is	New Classification is	Report Card Mark is	New Classification is	Report Card Mark is	New Classification is
100	100	*A*	90	*A*	82	*B*
95	98	*A*	88	*A*	80	*B*
90	95	*A*	85	*A*	78	*B*
85	92	*A*	83	*B*	76	*B*
80	90	*A*	81	*B*	75	*B*
75	89	*A*	79	*B*	70	*C*
70	87	*A*	77	*B*	68	*C*
65	85	*A*	75	*B*	65	*C*
60	80	*B*	70	*C*	60	*C*
55	75	*B*	65	*C*	55	*C*
50	65	*C*	55	*C*	50	*C*
45	55	*C*	50	*C*	45	*C*
40	40	*C*	40	*C*	40	*C*
35	35	*C*	35	*C*	35	*C*
30	30	*C*	30	*C*	30	*C*
25	25	*C*	25	*C*	25	*C*
20	20	*C*	20	*C*	20	*C*
15	15	*C*	15	*C*	15	*C*
10	10	*C*	10	*C*	10	*C*
5	5	*C*	5	*C*	5	*C*
0	0	*C*	0	*C*	0	*C*

The regrouping based on the results of the mid-term examination was checked up and corrected by teacher's judgment and class tests at the end of the second third of the term. The final examination was used as a basis for the final classification of the pupils for the next term's work.

So many changes of classification came about during the term that it was thought worth while to tabulate them and to study them.

CHANGES IN CLASSIFICATION

Periods of grouping	Groups No. of Pupils			Transfers					Percentages of distribution (Rough)		
	A	*B*	*C*	*A* to *B*	*B* to *A*	*B* to *C*	*C* to *B*	*A* to *C*	*A*	*B*	*C*
1. Initial—at end of 5 weeks	68	57	263	—	—	—	—	—	18	16	66
2. After Mid-term examination	53	120	221	22	9	16	58	1	13	31	56
3. After 2d third of term	74	78	236	10	17	35	5	1	19	20	61
4. Final—After Final examination	73	95	217	1	14	19	48	1	19	25	56

An examination of the above table discloses the following facts: 1. The teacher is inclined to underestimate the *C* pupils' abilities, probably on the assumption "once a *C* pupil, always a *C* pupil." This is seen from the fact that the percentage of the *C* group varies from 66 percent at the beginning of the term to 56 percent at the second regrouping, with a corresponding increase in the percentage of the higher groups.

2. As the term progresses and after the correction in the initial evaluation of the pupils on the basis of the mid-term examination, there is a noticeable change in the *B* group, some going to the *C* group and others rising to the *A* group. The first drop in the *A* group from 18 percent to 13 percent is due to the fact that in a few classes there were quite a number of "repeaters" who were put in the higher group because of a previous knowledge of the subject matter of the first five weeks. But as soon as real ability was tested on the mid-term examination, they could not live up to the expectations. These "repeaters" were transferred to the *B* group which shows a corresponding increase. The new students made a better showing rising from the *C* to the *B* group. The second drop from 31 percent to 20 percent in the *B* group is again due to the "repeaters" indicating that their previous training was aiding them less and less as the syllabus was increasing in difficulty.

3. The constant shifting from group to group with the general trend toward the higher groups shows the display of effort and interest on the part of the pupils. It also shows the ease with which transfers can be made from group to group a number of times during the term. This would be deemed rather difficult under a plan of classification where pupils of different abilities were segregated in different class-rooms.

4. The many transfers from one classification to another certainly impress the teachers with the fact that a class consists of dynamic rather than static personalities.

Classroom Procedure

The most essential prerequisite for the success of the plan is a definite classroom procedure. The following procedure was evolved in our school:

1. The new lesson taught took up the matter for which all the groups were responsible, *i.e.*, the work of the *C* group. The *A* and *B* pupils were to study their additional theorem assignment without the teacher's help in the classroom and were to solve their originals with hardly any help—the most they were entitled to was a suggestion. But since their position in the grouping depended upon their own individual efforts and originality, they were loath to expose their lack of ability and usually performed their tasks without help from the instructor.

2. The recitation of the home work of the previous day was conducted as follows: Only a few pupils were sent to the board from the *A* and *B* groups, the rest were from the *C* group. The recitation was usually started off by one of the *A* group. This was generally successful, and aroused a keen interest in the rest of the class.

3. To facilitate the conduct of the recitation with regard to the differentiated work and pupils, the *A*, *B* and *C* designations were used. These letters were written after the pupils' names in the rollbook; they were also written by each pupil at the board after his name and after the exercise to indicate the group to which the exercise belonged, since the *A* and *B* pupils were very often assigned board work from the *C* syllabus. This enabled the teacher to keep an accurate record of the calibre of each recitation if he cared to check minutely.

4. The written home-work assignment was treated in two ways. Part of it was handed in daily, the rest was kept in a notebook. The daily home work consisted of drill material or exercises of a simple nature. The *C* notebook included most of the theorems and the typical exercises of the *C* syllabus. The *A* and *B* notebooks contained all the theorems of the *A* and *B* groups only, and the more difficult exercises of all the groups.

5. The work was entered in the notebook only after it had been recited upon in the classroom. A student who was unable to assimilate a theorem or an exercise was given the opportunity of another recitation to overcome his difficulties before including the work in question in the notebook. This was to make sure that the work contained in it was correct. The notebooks were on certain announced dates checked by monitors, and occasionally, without previous announcement by the teacher.

Advantages and Difficulties of the Plan

This tri-grouping plan was used in all the beginning classes in geometry by teachers of experience as well as by teachers who taught the subject for the first time. While it can be easily surmised that the unexperienced teachers were somewhat burdened in the process of teaching three classes in one, they all ended the term expressing their enthusiasm for the plan. Since classroom procedure can be evolved only gradually, it must have been rather trying for a while to the new teachers. However, during the second term the plan was easily taken up. The difficulty of the class management involved in this plan was also offset to some extent by the daily lesson plans prepared for them.

There is no doubt that the teacher must be more alert to be able to take cognizance of the work of the three different groups but this is a matter of adaptation.

The all important question is, are the advantages and gains derived from the plan great enough to justify the additional effort? A consideration of the following points may furnish the answer.

1. With the introduction of the tri-grouping plan a new spirit seems to have been infused into the classes. A number of pupils rose within a short time, from a state of lack of interest, to the standard of the *A* group, and kept up at the head of the

class during the rest of the term. An *A* or *B* theorem or exercise, since it was known to the whole class to be purely a pupil product, drew the attention and interest of every one. This new spirit and the degree to which it was evinced in the classroom may be considered as the most important contribution of the plan.

2. The value of the plan lies in the fact that the bright pupils were given full opportunity for individual development of their abilities in presenting new theorems and difficult exercises. The recognition accorded by the whole class, as well as by the teacher, to the brighter pupil acted as an encouragement and incentive to the performer. The weaker pupil, on the other hand, was relieved of the burden of doing work beyond his capacity without being deprived of the opportunity of appreciating the good work done by the *A* and *B* pupils.

This method of grouping should not be confused with the grouping into "slow," "average" and "bright" pupils in separate classrooms. In the latter grouping we think of the "type" of pupils rather than of individuals. The pupil is placed in a class under the assumption that his abilities and interest in the subject have been absolutely determined for the rest of the term. As a matter of fact a great number of pupils in geometry do not grasp the subject at the beginning of the term. They get an understanding of it, and the enjoyment that comes with understanding, only as the term progresses. In the segregated grouping, however, no provision is made for the pupils who finally overcome their initial difficulty in the subject and become worthy of a higher classification.

Even if it were possible to transfer pupils during the term, they would find difficulty in adjusting themselves to a new environment with the brighter class already in advance of them. Such transfers are also undesirable on account of program difficulties.

Keeping the three parallel groups in the same room instead of different rooms and under different teachers has the following advantages:

1. It facilitates the transfer of pupils from one group to another from the point of view of program making.

2. There is greater emulation and also a spirit of self competition generated by the example set by the brighter pupils and the opportunity offered for promotion.

3. The *C* pupil is given the opportunity to come in contact with a wider knowledge of the subject; he is actually being taught by the brighter pupil. This contact enables the weaker pupil to learn the subject matter of a higher group with a view for promotion, a thing that can not be accomplished if the pupils are segregated.

4. The teacher is made more conscious of the individual pupil. The plan divides the class into smaller groups which insure closer "pupil-teacher" contact, since most of the pupils striving to better their standing come to the teacher for individual advice. Also since the calibre (*A*, *B* or *C*) of each pupil's work is noted by the teacher, the pupil becomes an individual in the eyes of the teacher.

5. The social setting in the tri-grouping of classes is nearer to real life conditions than the setting in which segregation of groups is practised. For, in life, no matter how poor our abilities are in any direction, we are, nevertheless, thrown into contact with others of better and even exceptional ability along the same lines. This contact creates in us a real appreciation of what good abilities are. This social setting is based on a positive psychology rather than on a negative one, in that it affords the pupil an example and an ample opportunity to better his standing, whereas under a plan of segregation of pupils of different abilities, this incentive is lacking for the slow pupil.

The syllabi for the different groups in the form of daily lesson assignments for the whole term attached to our plan presents a valuable feature of the Dalton plan, in that they enable the pupil to proceed ahead of the class whenever he finds it to his advantage to do so. However our plan, while it retains this important feature of the "individual" plans of instruction is at the same time not as laborious for the teacher and does not involve duplication of work, which is usually the case in the other individual plans.

It may be objected that the time taken up by the work of the *A* and *B* groups may result in a loss to the poorer students. But the fact is that *A* and *B* pupils were usually well prepared and took less than ten minutes going through their recitations without any waste of time. Besides, the psychological and material gains made during these few minutes is time well spent. Since the *A* and *B* work usually involved many steps which in-

volved the application of numerous geometrical facts, the pupils were asked for an oral outline of the proof before giving the detailed proof, thus affording the poorer pupils a valuable preview. While the reproduction of the proof of these exercises could hardly be expected of the *C* pupil, he gained greatly from listening to their discussion. He absorbed the fact knowledge as well as the methods of attack and analysis for which he is responsible in his own work. The recitations of the *A* and *B* pupils were an ever-recurring example of work brilliantly done.

The plan is being retained in all the beginner's classes in geometry this term and is being tried as an experiment in one class which is preparing for the Regents' examination. As we become more experienced with it, we hope to use the *A*-*B*-*C* plan in all of our mathematics classes.[1]

[1] When this paper was ready to be submitted my attention was called to an article on the same subject in the MATHEMATICS TEACHER for November, 1928, by Mr. Fletcher Durell. The similarity between the conclusions drawn in both papers is remarkable. In view of this similarity, would it be reasonable to assume that any attempt to provide for individual needs in pupils would necessarily lead to a plan closely resembling that of Mr. Durell and the one here described?

The Association of Mathematics Teachers of New Jersey held their 32d regular meeting at the Atlantic City High School on Monday, November 12th, at 2 P.M., with Miss Josephine Emerson, of Kent Place School of Summit, N. J., presiding. The following program was given: "What Constitutes Relative Difficulty in Algebra," Andrew S. Hegeman, Central High School, Newark; "Mathematics and Religion," Harrison E. Webb, Market Street High School, Newark; "Unrealized Possibilities in Junior High School Mathematics," Miss Vera Sanford, Columbia University.

CONCERNING ORIENTATION AND APPLICATION IN GEOMETRY

By D. G. ZIEGLER

Teachers College, Columbia University

Every teacher of geometry must answer in one way or another two questions that are certain to arise in the minds of his pupils early in the course. On the answer given to these questions will rest largely the success or failure of the pupil to profit by his experience with geometry.

What seems to be the vital problem for the teacher to face early in the course is that of properly adjusting the pupil to new lines of thought and new methods of thinking. This problem comes up as soon as he asks the question "What is geometry, anyway?" Suppose that a pupil comes to you and asks that question, what would your answer be? His question is important because of its implications, and because of the problem it raises.

The problem is one that concerns the pupil and his relation to his past experiences, and one whose solution lies in his understanding and appreciating the meaning of some of them.

We all know that by the time a child is ready to make his debut into a class in geometry he has unconsciously made a number of geometrical discoveries. The infant has shown an interest in "size," "shape," and "position" from the earliest days. Very early in life a child discovers rather intuitively that it is shorter to 'cut corners' than to go around. In his several games, he has shown a knowledge of the various geometrical figures. The child who has just learned successfully to climb a flight of steps has made a geometrical discovery valuable to his future experience; and the little girl who has successfully completed her first doll dress has made yet other important discoveries. We could go on and mention other cases, but the point is that every child has accumulated a vast number of experiences from which he has learned intuitively a number of geometrical truths. These experiences constitute the teacher's opportunity.

Keeping in mind that the pupil has a question which to him

at least is important and to which he expects a clarifying answer, what is the teacher's function? It seems obvious that the teacher must seize upon these child experiences and interpret them to him in the light of his present activities and his future experiences. It seems evident that if we expect to help the pupil get his bearings, maintain his balance in a strange sphere, to become oriented we might say, we must show him that he already has a knowledge of the nature of geometry. Not only must we make him conscious of his intuitive knowledge, but we must show him how it has fitted into his past experience, how the process of experience is a continuous one, and how *his* future as well as that of *everyone* is vitally affected by geometric truths. In this idea, I believe there lies one possible answer to the question, "What is geometry, anyway?"

I remember well my first day in a geometry class and the attempt made by the instructor to tie up some of our past experiences to geometry. "Look out of the window and all about, you will see geometry. (I did and saw swaying tree branches.) There, you see the siding on that house, that is an example of parallel lines. And the limbs of the trees and the telephone poles, they are fair examples of cylinders. So throughout nature and all about, you can see geometry. Now turn to page one, theorem one."

This bit of explanation, I suppose, was meant to orient the class in the study of geometry, to arouse the intuitive background in the minds of the class. It may have borne results but certainly to only a very limited degree, for as we plunged into the formal work of the course no other attempt was made to bring out intuitive knowledge and I soon found myself in that inquiring state which asks, "What is it all about?"

The second question that is certain to arise and disturb the teacher's complacency need hardly be stated, it is so common. It is an age-old question. Consider the "Ancient Greek Teacher" whose pupil asked, "What will I profit by the study of geometry!" It is a question characteristic of pupils even today, and a question to which they have a right to an intelligent answer. The Ancient Greek Teacher squelched *his* pupil by tossing to him a coin. That was his answer to the question.

The pupil's query as to "why study geometry?" is of too much significance to be put down thoughtlessly or ruthlessly.

It is related too closely with the first question. It implies an ignorance of what geometry is. It shows that the pupil does not understand or appreciate his past experience with geometrical truths and their relation to his present and future needs. The problem is one of understanding and appreciation.

In securing understanding and appreciation, I believe that applications can be made to play a strong part. If the pupil is shown a great number and range of examples wherein geometry plays an important part, and these applications are evaluated in the light of human needs, then the problem is nearing solution. For example, consider the difference between presenting the work on congruence directly from the text and by using applications. In the first case the idea is developed in a formal way with superficial results. In the second case it is developed by the pupils seeing many cases of congruence in operation in modern industry, and by giving them an opportunity to pass judgment on the value of some of the industrial products to modern life and needs. It seems that this emphasis in presentation is more promising of fruitful results. This does not mean that all formal work should be omitted, but the essential formal work should be introduced by means of many applications.

Summing up the point of view that has been presented thus far; if the pupil finds his bearings, if he becomes conscious that he already possesses much geometrical knowledge, and if he can come to an appreciation of geometry through many applications, then he will see geometry in its true setting, and there will be less to fear from the formal work that follows.

The remaining part of this paper deals more concretely with the above ideas of orientation and application as worked out in my own classes. The plan that was used in the course is briefly sketched in the following paragraph.

The first six weeks were taken up with a course in intuitive geometry, during which time the text was not referred to, nor was formal work begun. During this six weeks' course and continuing throughout the rest of the year, the pupils were instilled with a spirit of investigation for applications of geometry. In addition to this, a number of projects were put into the program from time to time so that the pupils might have practice in making applications as well as in detecting them.

The following are some of the points of work taken up in the intuitive course.

1. Ancient need for a reliable method of restoring property lines.
2. The evident meaning of the word geometry.
3. Methods and difficulties of direct measurement.
4. Thales' introduction of triangulation, the indirect method of measurement, and its value to modern needs.
5. Introduction of the idea of congruence, and its function in modern industry.
6. Introduction of the idea of symmetry, and its uses today.
7. Project: Search for and analysis of geometrical figures as found in modern industry and as used in life in general.
8. Transition from intuitive to formal work.

Under 1 it was brought out that the overflowing of the Nile raised a very important problem of redetermining property lines.

Following out this thought, it was shown in 2 that the science or method devised to meet the difficulty took its name from the very nature of the difficulty, earth measurement.

Number 3 dealt with the difficulties of using direct measurement, by showing that "inaccessible distances" cannot be measured directly.

The fourth point showed that we are not dependent on the direct method of measurement. It was pointed out how Thales used the triangle as the basis for another system of measurement, thereby initiating a reform in measurement which has had a profound effect on all life since that time. Examples of the use of the indirect system were mapped out to demonstrate its value to modern scientific work. The pupils were also given an opportunity to use the indirect system in a project to be described later.

The idea of congruence, its function in modern industry, and the methods of detecting congruence were next taken up. In connection with this subject, the pupils listed many of the skills, trades, occupations, and industries that made use of the principle of congruence. They then listed as many of the products of industry as they could that were illustrations of congruence.

In introducing the idea of symmetry, "ink-blot" figures or

"gobble-inks" were used. This proved to be a highly interesting exercise to the pupils because of the variety of grotesque figures produced. This exercise proved to be stimulating to the imagination because almost at once the pupils began to see in the figures things that resembled other things. Following this introduction, much the same treatment was given symmetry as was given congruence, the purpose being to demonstrate the uses and importance of symmetry today.

Number seven deals with an idea worked out in the course. This idea proved to be very fruitful from the standpoint of the questions discussed in the first part of this paper. The pupils were sent on trips of investigation to find and reproduce illustrations where geometrical figures were used. The boys went to the clothing stores and copied all the latest designs in men's ties, while the girls visited the department stores and secured samples of various kinds of textiles. Besides, both boys and girls copied lace curtain and linoleum designs. The samples and sketches were then discussed and the designs analyzed for geometric figures. This finished, the specimens were mounted on cardboard with a description in the pupil's own words. The mounted samples were saved and used at the end of the course in our mathematical exhibit.

In connection with this piece of work, a short period was given to discussion of some of the ways in which people unconsciously use geometry. The discussion was worthwhile as it led to the conclusion that geometry enters into practically every phase of life, and that considerations of "size," "shape," and "position" are things that cannot be ruled out.

From this point a shift was made from intuitive to formal work. In the transitory work, the assumption on which geometry rests were taken up. Along with this were taken up some of the statements that can be accepted without proof, things that every child knows to be true. For example, the theorem, *If two lines (straight) intersect the vertical angles formed are equal*, was presented, examined and illustrated but no attempt was made at formal proof. Every child knows that it is true. He "feels it in his bones." Congruence was again touched upon in connection with "triangles." The three methods of determining congruence (*asa, sas, and sss*), were discussed and illustrated but no attempt was made at formal proof. Parallel

lines were next discussed. The properties of parallel lines and the relationship that exists between the angles when the lines are cut by transversals were given considerable treatment. This study showed clearly that it is unnecessary to prove some of the theorems concerning parallel lines and their transversals. Finally the work swung over to the method of formal proof. The need for scientific thought and arrangement in the work was made clear, and then formal proof was gotten under way.

From the standpoint of getting results and of the time element, the six weeks' course in intuitive work was a success. It resulted in a decided growth on the part of the pupils in attitudes, appreciation, and understanding. Concerning the time element, the six weeks spent in the beginning did not cramp the rest of the course for time. By the end of the course we were almost two weeks ahead of where we were the preceding year.

In the concluding part of this paper, mention is made of some of the things that were done in order to tie up the more formal work of the course to something practical. A real effort was made to give the pupils an understanding of practical applications.

First is the program of planned field trips, a method devised to encourage the pupils to look for and make applications whenever they could. One of these field trips had for its object: To extend a straight line through an obstruction (a hill). The line was extended, without the use of instruments, through a large field from street to street perpendicular to one of them, thus forming a rectangle. The distances from both ends of the new line to a parallel street were measured to check up on the accuracy of the work.

Another trip was taken when the work on areas was begun. Here an interesting development took place. The class became interested in the city water tank. The circumference was measured and the radius and base area computed. The height was measured two ways, first, by multiplying the number of steel plates by the height of one plate; second, by measuring shadows and using similar triangles. We then began to determine lateral area, total area, and volume both in cubic feet and in gallons. Thus, the interest of the class took them outside the field of plane geometry. It was a perfectly natural development of

interest, and a case where one should not insist on maintaining the hard and fast lines of division between subjects.

Another type of practical work was taken up in connection with the work on ratio and proportion. The purpose of this was to explain scale drawings, and to show how the figures could be computed and the drawings made.

Each pupil was asked to measure up one wall of a room in his home, measuring also the dimensions of furniture, pictures, and windows that were in that elevation. These measurements were placed in a table, a convenient ratio was taken, and corresponding scale measurements were computed and placed in the table. Then the scale drawing was made. The pupils kept their work on cardboard, and this piece of work as well as the others figured in the mathematics exhibit.

The crowning event in the course was the mathematics exhibit which was placed with a general school exhibit. Notebooks were laid out showing the different phases of the work that had been covered. Educational posters intended to answer the same old questions of, "What is geometry?" and "What good is it?" were placed about the room. The mounted records mentioned previously completed the exhibit. The pupils took a great pride in the exhibit, and did much of the work in getting it ready and organized. At each session of the exhibit, a committee of pupils were on hand to escort the visitors through and to explain the work that they had done. Others of the class brought their friends and parents to the mathematics room so that they would not miss the exhibit. This improved the attitude of the pupils towards their work in geometry.

Perhaps the most important thing to look for in any course is the attitude that the pupils take towards their work. It is important because attitudes of mind certainly have some bearing on how much and what kind of learning is going on.

At first an aversion on the part of four girls was noticed. They stoutly maintained that geometry was all right as a required subject for boys, but that it should be an elective for girls as it was of no use to girls. This case was followed up with much interest in what the final outcome would be. As the intuitive work progressed, and the work became more and more interesting, opposition on the part of two of the girls ceased to exist. The two other girls, due to reasons beyond the instruc-

tor's control, never gained a favorable attitude towards their work. Considering the class as a whole, a lively interest existed as soon as the intuitive course got under way. This interest seemed to grow as the course progressed. The spirit shown by the class in connection with the exhibit, and the spirit shown when the class voted to improve the classroom by leaving behind them an appropriate picture as a memorial of the class of 1925–26, was indicative of the attitude held by the class.

The increased measure of success this year over the preceding year is due to three things: increase of experience on the part of the instructor, *The Intuitive Course*, and the emphasis placed on application. The first was no doubt a factor, but the intuitive course, I believe, was the biggest contributing factor to the increased measure of success experienced this year. Through the intuitive work the natural background of the pupils was brought to bear in preparation for the more formal work. It was this factor that oriented the class and helped them to get their bearings in a strange sphere of thought. It proved to be valuable in producing greater results, and also a time saver instead of a time waster. The emphasis placed on application produced desirable results, as it gave the pupils an understanding of how geometrical truths are applied in the process of living. It gave them practice in making applications as well as in detecting them.

The intuitive course was used in meeting the question, "What is geometry?" and an attempt was made to answer the question, "Why should I study geometry?" by using many applications. The foregoing is offered not as the only solution to the above problems, but only as a method that has been tried out and found successful in practice.

ORDER THE FOURTH YEARBOOK NOW!

RADIO AND AIRPLANE DISTANCES

By P. H. NYGAARD

North Central High School, Spokane, Washington

With the recent development of the radio and the airplane considerable interest has arisen in the problem of determining the shortest distance between any two points on the earth's surface. The writer proposes in the following discussion to treat the problem in a manner mathematically exact, and at the same time make the explanation simple enough to be understood by any high school student who has had experience with elementary problems involving sines and cosines. It is hoped that teachers may find the discussion helpful in providing supplementary work for high school mathematics classes, and for reference reading on the part of interested students.

Nearly everyone who has followed the recent exploits of long distance aviation has heard of the term, "great circle course." Few people, however, understand what it means. Not many know how to calculate in miles the length of such a course between two points on the earth's surface. For example, an aviator wishing to make the shortest possible non-stop flight from northern Maine to Spokane would not fly straight west, although Spokane is almost due west from northern Maine. Paradoxical as it may seem, the shortest path would be to go 19° north of west, and then gradually change the course toward the south. At the half way point the direction would be approximately due west. Beyond that it would be south of west, and coming into Spokane the direction of flight would be about 18° south of west. The point farthest north would be a little north of Winnipeg, about 160 miles further north than Spokane and about 180 miles further north than Maine. The path described is the *great circle course*. The distance would be about 2,212 statute miles. Had the plane been flown straight west all the way, the distance would have been about 2,248 statute miles. The difference in the distances would have been much more pronounced if the points had been farther apart or if they had been located farther north.

We shall now explain what is meant, mathematically, by a great circle course. It can be proved that the shortest distance between two points on the surface of a sphere is along a circle drawn through the two points and having for its center the center of the sphere. Such circles are called *great circles* to distinguish them from other circles whose centers are not at the center of the sphere. The earth is so nearly spherical in shape that it can be considered to be a sphere without introducing appreciable errors. The earth's equator and all the meridians are great circles, as are any circles that divide its surface into two equal parts. The arctic circle, for instance, is not a great circle, because its center is quite near the north pole on the straight line connecting the pole to the earth's center. None of the parallels of latitude found on a globe, except the equator, are great circles. The best way to realize the facts about great circle courses is to take a globe or other round object and then stretch a string between any two points on the surface. The string will form the shortest (great circle) course between the points.

The calculation of the *great circle distance* between two points can be easily made if their latitudes and longitudes are known. Let A be the latitude of one point and B that of the other. Denote the difference in their longitudes by D. Let X be the number of degrees along the great circle between the two points. Then it can be proved by spherical trigonometry that

$$\cos X = (\cos A \cdot \cos B \cdot \cos D) + (\sin A \cdot \sin B).$$

By means of this formula and a table giving sines and cosines of angles up to 90°, the number of degrees in X can be easily found. Since each degree equals about 69.1 statute miles, we multiply the number of degrees in X by 69.1 to get the great circle distance in terms of statute miles.

Suppose a radio listener in Los Angeles has heard a broadcasting station in New York City and wishes to find the distance. New York is at 39° 45′ *north latitude* and 74° *west longitude*. Los Angeles is at 34° 03′ *north latitude* and 118° 15′ *west longitude*. Thus, $A = 39° 45'$, $B = 34° 03'$, and $D = 44° 15'$. Substituting these values in the formula, $\cos X = (\cos 39° 45' \cdot \cos 34° 03' \cdot \cos 44° 15') + (\sin 39° 45' \cdot \sin 34° 03')$. We shall use for the solution a three place table giving sines and cosines of degrees only, and shall estimate for the minutes. We find cos

39° 45′ to lie ¾ of the distance between .777 and .766, so cos 39° 45′ = .769. Similarly, cos 34° 03′ = .829, cos 44° 15′ = .716, sin 39° 45′ = .640, and sin 34° 03′ = .560. Hence cos X = (.769·.829·.716) + (.640·.560) = .815. We find X by seeing how many degrees there are in the angle whose cosine is .815, getting X = 35.4°. Distance = 35.4 × 69.1 = 2,455 statute miles. By using a more accurate table we obtain 2,458 statute miles.

As long as A and B are north latitudes and D does not exceed 90°, there is nothing complicated in the method. The formula holds, however, for all latitudes and longitudes. All that it is necessary to know is how to find the sines and cosines of angles larger than 90° and of negative angles. Anyone having had a course in plane trigonometry will understand what to do. For the benefit of others, sufficient explanation will be given so that the formula may be used.

Either or both of the latitudes may be south; then A or B, or both, will be negative, as the case may be. The cosines of negative angles are simple, for cos $(-A)$ = cos A and cos $(-B)$ = cos B—that is, the cosines are the same whether the angles are negative or positive. The sines are not so easy. Sin $(-A)$ = −sin A and sin $(-B$ = −sin B—that is, the sine of a negative angle is the same numerically as the sine of the positive angle, but it must have the negative sign prefixed. The difference in longitude, D, need never be negative, but may have values up to 180°. If D lies between 90° and 180°, then cos D = − cos $(180° - D)$—that is, subtract D from 180°, look up the cosine of the angle thus obtained and prefix the minus sign. Several of the quantities substituted in the formula may then be negative, and, if so, due consideration must be given to the algebraic rules for multiplying and adding positive and negative numbers. If cos X turns out negative, X will be between 90° and 180°. In this case find the angle, less than 90°, whose cosine is the same numerically as the value obtained for cos X. Then to get the number of degrees in X subtract this angle from 180°.

Let us now take a problem as difficult as possible. A steamer going from Yokohama in Japan to Valparaiso in Chile would follow a great circle path, for there is no intervening land to block the course. We shall proceed to calculate the distance. Yokohama is at about 35° 25′ north latitude and 139° 35′ east

longitude. Valparaiso is at about 33° south latitude and 71° 30′ west longitude. Here $A = 35°\ 25'$, $B = -33°$, and $D = 148°\ 55'$. Hence cos $X = [\cos 35°\ 25' \cos(-35°) \cos 148°\ 55'] + [\sin 35°\ 25' \sin(-33°)]$. We shall again use a three place table giving values for degrees only. Cos 35° 25′ lies 5/12 of the distance between .819 and .809, so cos 35° 25′ = .815. Cos (−33°) = cos 33° = .839. Cos 148° 55′ = − cos (180° − 148° 55′) = − cos 31° 05′ = − .856. Sin 35° 25′ = .580. Sin (−33°) = − sin 33° = − .545. Hence cos X = (.815 · .839 = − .856) + (.580 · − .545) = (− .5853) + (− .3161) = − .901. Now, find the angle whose cosine is + .901. It is 25.7°. $X = 180° - 25.7° = 154.3°$. Distance = 154.3 × 69.1 = 10,660 statute miles. The actual distance would be a few miles more, because a correction should be added for the deviations necessary to get out of and into the harbors.

NEWS NOTES

The Mathematics Section of the Central Association of Science and Mathematics Teachers held its annual meeting at the Little Theatre in the Reynold's Club at the University of Chicago on Friday afternoon, November 30th, 1928. The officers of the section were: Edwin W. Schreiber, Chairman, University of Michigan, Ann Arbor, Mich.; Martha Hildebrandt, Vice-Chairman, Proviso High School, Maywood, Ill.; Margaret Dady, Secretary, Waukegan High School, Waukegan, Ill.

The following program was given:

"The Drift in Junior High School Mathematics," Mary A. Potter, Racine High School, Racine, Wis. Discussion led by Martha Hildebrandt, Proviso High School, Maywood, Ill.

"Worthwhile Literature in the History of Mathematics Desirable for the Teacher in Secondary Schools," Prof. Louis C. Karpinski, University of Michigan, Ann Arbor, Mich.

"Mathematics in Industry," Clarence J. Leonard, Cass Technical High School, Detroit, Mich. Discussion led by Marx Holt, Principal, Seward School, Chicago, Ill.

This is an important group of teachers of mathematics in the Middle West who for a long time have been aiding in the development of better methods of teaching of mathematics. The magazine of the Central Association, *School Science and Mathematics*, is now under new management and is carrying on the work so well begun by the pioneers in the organization.

The MATHEMATICS TEACHER takes this opportunity to wish the new management the greatest success.

ANNUAL MEETING OF THE NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS

Hotel Statler, Cleveland, Ohio, February 22 and 23, 1929

PROGRAM

FRIDAY, 10:00 A.M. Joint meeting of the Directors and the local committee.

FRIDAY, 12:30. Directors' luncheon.

FRIDAY, 2:30 P.M. Annual business meeting.

FRIDAY, 7:45 P.M. Open meeting.

Address of Welcome:
C. H. LAKE, Assistant Superintendent of Schools, Cleveland.

Greetings from the Cleveland Mathematics Club:
FRED N. BURROUGHS, President.

Response for the Council:
MARY S. SABIN, Second Vice-President, Denver, Colorado.

Address: The Permanent Nature of the Necessity for Mathematics in the Secondary School:
LOUIS C. KARPINSKI, University of Michigan.

SATURDAY, 9:00 A.M.

Current Tendencies in Mathematical Teaching. A worldwide survey based on the Fourth Yearbook:
VERA SANFORD, The Lincoln School of Teachers College.

Mathematics and the Future:
CHARLES N. MOORE, University of Cincinnati.

For the Good of the Council:
W. D. REEVE, Teachers College, Columbia University.

SATURDAY, 2:00 P.M.

Mathematics Plus:
FLORENCE BROOKS MILLER, Fairmount Junior High School, Cleveland.

The Cultural Value of Mathematics:
W. W. RANKIN, Duke University, Durham, North Carolina.

The Place of Mathematics in Junior High School Edution:
AGNES GRANT ROWLANDS, Jamaica Training School and Hunter College, New York City.

Teaching Geometry into its Rightful Place:
J. O. HASSLER, University of Oklahoma.

SATURDAY, 6:00 P.M. Banquet.

Greetings from guests of honor.

Address: Informational Mathematics Versus Computational Mathematics:
CHARLES H. JUDD, School of Education, University of Chicago.

Order banquet tickets from Fred N. Burroughs, John Adams High School, Cleveland, Ohio. Probable price $3.00.

For official blanks for reservations at the Statler write Harry C. Barber, Exeter, N. H.

We believe that our readers will be interested in the following program of the National Society of College Teachers of Education to be held in Cleveland, Ohio, February 25–27, 1929:

MONDAY FEBRUARY 25, 1929

Meeting for Members

9:15 A.M., Georgian Room, Hotel Cleveland.

General Topic: "Fundamental Courses in Teacher Training"

1. "A First Course in Education" (20 minutes), C. B. Gentry, Connecticut Agricultural College.
2. "Function of the Introductory Course in Education" (15 minutes), Arthur J. Jones, University of Pennsylvania.
3. "Professionalizing Psychology for the Elementary Teacher in Training" (15 minutes), Bessie L. Gambrill, Yale University.
4. "Directing Research of Teachers in Service" (15 minutes), C. G. F. Franzen, University of Indiana.
5. Discussion and Conclusions (20 minutes), Henry W. Holmes, Harvard University.

Open Meeting,

2:00 P.M., Georgian Room, Hotel Cleveland

Central Theme: "Improvement of some Phases of Instruction and Administration in Colleges of Education."

1. Announcement by the Committee on Sate Histories of Education, Stuart G. Noble, Tulane University, New Orleans.
2. "Problem of Negro Education in Northern and Border Cities" (20 minutes), Louis A. Pechstein, University of Cincinnati, Cincinnati.
3. "The Improvement of Institutional Processes as a Focus of School Training" (20 minutes), Walter R. Smith, University of Kansas, Lawrence.
4. "The Practical Necessity and the Validity of the Philosophy of Education" (20 minutes), J. H. Coursault, University of Missouri.
5. "Criteria for Doctor's Dissertations" (20 minutes), Charles H. Judd, University of Chicago.

TUESDAY, FEBRUARY 26

Meeting for Members, Round Tables, 9:15 A.M., Hotel Cleveland.

1. Educational Psychology—Room H.
2. History of Education—Room D.
3. Philosophy of Education—Room R.
4. Educational Sociology—Rooms E and F.

2:00 P.M.

Program for Open and Joint Session of the National Society of College Teachers of Education and the Educational Research Association—Public Auditorium Music Hall, South Wing.

1. "How Historical Knowledge Enters into Thinking about Educational Problems," Edward H. Reisner, Teachers College, Columbia University.
2. "An Attempted Analysis of Factors in College Success," Herbert A. Toops, Ohio State University.
3. "Teaching Load in State and Other Universities," Fred C. Ayer, University of Texas.

4. "Some Experiments in Methods of Instruction in College and University Subjects," Harl A. Douglas, University of Oregon.

WEDNESDAY, FEBRUARY 27, 1929, 9:15 A.M.

9:15 A.M., *Meeting for Members*, Georgian Room, Hotel Cleveland. Topic: The 1929 Yearbook

1. "Presentation of the 1929 Yearbook by the Chairman of the Yearbook Committee" (30 minutes), William S. Gray, University of Chicago.
2. "Constructive Activities in Improving Instruction in eighty-seven Institutions" (25 minutes), Floyd W. Reeves, University of Kentucky.
3. "Instructional Resources Available for Collegiate Educational Research" (25 minutes), Melvin E. Haggerty, University of Minnesota.
4. "Major Problems in the Improvement of Instruction in Higher Institutions" (20 minutes), George F. Zook, University of Akron.
5. Discussion from the floor.

OFFICIAL BALLOT

For Election of Officers at the February 1929 Meeting of the

NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS

FOR SECOND VICE-PRESIDENT

(VOTE FOR ONE)

HALLIE S. POOLE, LAFAYETTE H. S., BUFFALO, N. Y. ☐

CHARLES S. STONE, UNIVERSITY H. S., UNIVERSITY OF CHICAGO ☐

FOR MEMBERS OF BOARD OF DIRECTORS

(VOTE FOR THREE)

JOHN R. CLARK, NEW YORK CITY - - - ☐
ALFRED DAVIS, ST. LOUIS, MO. - - - - ☐
ELIZABETH DICE, DALLAS, TEX. - - - - ☐
LAURA FARNUM, MINNEAPOLIS, MINN. - - ☐
J. O. HASSLER, NORMAN, OKLA. - - - - ☐
E. W. SCHREIBER, MAYWOOD, ILL. - - - ☐

Please mark this Ballot at once, and mail it to J. A. FOBERG, *Secretary*, STATE TEACHERS COLLEGE, CALIFORNIA, PENNA.

NEW BOOKS

Ueberlick über die Geschichte der Elementarmathematik. W. LIETZMANN. 2d ed., Leipzig, B. G. Teubner, 1928. Pp. viii + 80. Price 2 RM.

Aus der Mathematik der Alten. W. LIETZMANN. *Ibid.*, 1928. Pp. viii + 66. Price 1.80 RM.

In the preceding number of the MATHEMATICS TEACHER attention was called to one of the several series of small handbooks for teachers and students now being published in Germany. The two books mentioned above are from another and no less important series published by the well-known firm of Teubner, in Leipzig. This series consists of supplementary teaching material issued by Dr. Lietzmann of the Oberrealschule at Göttingen, one of the most active of the German contributors to the success of the International Commission on the Teaching of Mathematics, and the author of various important reports and works on the improvement of secondary mathematics in his country.

The first of the above-mentioned handbooks appeared some time since, but it is now published in larger form and mechanically in better dress. Its purpose is to show the educational value of illuminating the work in elementary mathematics by the use of historical material. To this end the author begins by giving a general view of the development of the subject, considering (1) the prehistoric and early historic period, (2) the Greek contributions, (3) the middle ages, and (4) modern times. He then proceeds to a consideration of eight subdivisions of elementary mathematics, as follows: (1) Computation; (2) "Arithmetik," this word in German having a different meaning from our "arithmetic," and covering the number systems found in algebra, the algebraic operations, roots, logarithms, series, and some elementary work in probability; (3) Algebra, using the word in the historic sense which still obtains in various European countries—the work in equations; (4) Planimetry; (5) Stereometry; (6) Trigonometry; (7) Analysis, including analytic geometry, functional relations, and elementary calculus; (8) Conics, treated both synthetically and analytically. The work is throughout abundantly supplied with illustrations from ancient monuments and from early printed books. There is not, in any language, a better primer of the history of elementary mathematics, and for the price (48 cents in paper binding, in Germany) there cannot, under present conditions, be published in the English-speaking countries anything so good.

The second of the works above mentioned is a source book containing extracts referring to elementary mathematics. This differs in two important respects from the one by Dr. Wieleitner, mentioned in the preceding number of this periodical. In the first place the extracts are substantially all from Greek sources, as the title would naturally suggest —the chief exception being a brief reference to the work of Ahmes (Rhind Papyrus). In the second place the author usually gives merely the source material, modified only

by the use of modern algebraic symbols, a plan which, by many readers, will be preferred to that of Dr. Wieleitner, who frequently intersperses discussions as to the meaning and significance of the extracts. The plan of giving the source material by itself is likely to appeal to one who has some mathematical training, while the plan of interspersing the comments will be more favored by those whose mathematical ability is more limited.

A list of the chief selections will be helpful to those in search of such material. Briefly stated, they are as follows: (1) The nature of pure mathematics—Plato; (2) The establishing of the system of natural numbers—Archimedes; (3) Equations with rational, positive roots—Diophantus; (4) Pythagorean numbers—Diophantus; (5) Foundations of geometry—Euclid, Heron; (6) The Pythagorean theorem—Plato, Euclid; (7) Heron's formula for the area of a triangle; (8) The construction of the regular decagon and pentagon—Heron; (9) Attempts at the quadrature of the circle—Simplicius; (10) Harmonic points and pencils—Pappus; (11) Euclid's treatment of the volumes of triangular pyramids of equal heights; (12) Archimedes on the volume of a sphere; (13) The three conics—Apollonius; (14) Approximation methods in measurement—Heron; (15) The diopter applied to surveying—Heron; (16) The form of the earth—Ptolemy; (17) The ecliptic and geographical latitude—Ptolemy.

The work is well illustrated and in several cases the Greek sources have been supplemented by extracts from various modern writers, including Fermat, Schwenter, and Byrne.

Altogether, these two manuals by Dr. Lietzmann are perhaps the most helpful ones he has yet prepared—and this is to give them high praise.

DAVID EUGENE SMITH

The History and Significance of Certain Standard Problems in Algebra. By VERA SANFORD. New York, Bureau of Publications, Teachers College, Columbia University, 1927. vii + 102 pages. Price $1.50.

As stated by the author, "The purpose of this essay is to consider briefly the reasons for the use of verbal problems in textbooks in elementary algebra; to study the history of certain typical problems and determine the factors that make them transitory or make them lasting; and finally to evaluate the place of problems in our present curriculum." And "a problem was considered as algebraic if it is one which we would solve by algebra today."

This study with few exceptions is made from original works. The problems quoted are taken directly from early arithmetic and algebra textbooks. These are frequently given in footnotes but sometimes in the body of the text where translations are also found. The conclusions drawn are therefore convincing. There is real authority here for students of the history of mathematics and, what is far more important, for teachers of algebra. For this piece of research teaches many a lesson that needs to be learned. Among these are that "to be lasting a problem must be independent of local conditions and must involve concepts that are the common property of thinking individuals" (p. 76); and again that "genuine problems will survive so long as mathematics seeks to bind

itself to the business, economic and scientific worlds. But in the future as in the past, the problems that are the best will be those that are real and important in the eyes of the students" (p. 95).

The work is replete with valuable suggestions. Among these are that: "Present day interest in bringing the topic of logarithms into elementary mathematics should result in the including of this [annuity] group of problems also" (p. 46); that: "we use such colorful problems [ones based on progressions found in many of the texts] rather than the hackneyed bouncing ball of the usual textbooks" (p. 66). To these, others could be added because there seems to be no element of present day problem content and methods of solution that is not discussed.

The purely historical phase of this study may well be stressed. Much interest attaches to such matters as the shift of problems from arithmetic to algebra through the invention of symbolism which made the equation possible; to the effect that the invention of logarithms has in shifting problems that concern the use of inverse cases of the compound interest law from arithmetic to higher algebra; and, to the conclusion that additions are made relatively soon after the need for them is apparent but eliminations are apt to be tardy.

The style of the work is clean cut with a frequent turn of a phrase that lightens an entire page. We illustrate by a transcription of one problem as follows:

> "Champenois made an illustration of the power of a geometric progression particularly impressive when he invented the story of the learned but poverty stricken Master of Arts who offered a tailor 4,888 livres diminished twenty times by one half for a robe that the tailor valued at 58 livres. The tailor accepted the offer and the scholar prudently took the robe and then showed the tailor that the price came to a trifle over twopence. He gave the tailor threepence and asked for the change. The tailor, who had lost a great deal by the transaction already, offered to gamble with him for it and lost. The moral of it is that the scholar got himself a fine robe by means of his knowledge of a rule of arithmetic."

But let the reader go to this book for a few hours of real delight.

The contents are: Why Problems are Taught, Efforts to Make Problems Interesting, Methods of Solving Problems, Commercial Problems, Genuine Problems from Scientific Mathematics, Recreations, Pseudo-real Problems in Present-Day Textbooks, Changes in Problem Material, and The Future of Problems. The summary at the end of each chapter is an excellent feature of the work. It is unusual to find any book printed in this country without an index. The usefulness of this history as a book of reference would have been increased had one been included.

LAO G. SIMONS